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OF NOTED MATHEMATICIANS, ETC.

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ON A CERTAIN RATIONAL CUBIC TRANSFORMATION IN SPACE.*

By DR. M. W. HASKELL.

Through any point A in space can be drawn one and only one line meeting a given twisted cubic in two points. If B be the harmonic conjugate of A, the relation between the coördinates of A and B must be given by a birational transformation, and this transformation has some interesting properties.

Let the homogeneous coördinates of A be $a_0 : a_1 : a_2 : a_3$, those of B be $b_0 : b_1 : b_2 : b_3$, and let the twisted cubic be given parametrically by the formulae

$$x_0 : x_1 : x_2 : x_3 = \lambda^3 : \lambda^2 : \lambda : 1.$$

The coördinates of any point on the line AB will be of the form

$$\mu a_0 + \nu b_0 : \mu a_1 + \nu b_1 : \mu a_2 + \nu b_2 : \mu a_3 + \nu b_3 ;$$

and, if this line is to meet the twisted cubic at all, the three equations

$$(1)\ldots\ \left\{\begin{aligned} (\mu a_0+\nu b_0)(\mu a_2+\nu b_2)-(\mu a_1+\nu b_1)^2&=0\\ (\nu a_0+\nu b_0)(\mu a_3+\nu b_3)-(\mu a_1+\nu b_1)(\mu a_2+\nu b_2)&=0\\ (\mu a_1+\nu b_1)(\mu a_3+\nu b_3)-(\mu a_2+\nu b_2)^2&=0\end{aligned}\right.$$

* Presented to the American Mathematical Society at its Chicago meeting, January 2, 1902.

must have a common root. If the line AB meet the cubic in two points, these equations must be identical and their coefficients proportional; and finally, if the points of intersection divide AB harmonically, the values of $\mu : \nu$ given by these equations must be equal and opposite and the coefficients of $\mu\nu$ must vanish:

$$(2)\dots\quad \begin{cases} a_2 b_0 - 2a_1 b_1 + a_0 b_2 = 0 \\ a_3 b_0 - a_2 b_1 - a_1 b_2 + a_0 b_3 = 0 \\ \qquad a_3 b_1 - 2a_2 b_2 + a_1 b_3 = 0 \end{cases}$$

These three equations are sufficient to determine the coördinates of either point in terms of those of the other, and their symmetry shows that the transformation inverts into itself, which is also evident from the geometrical conditions. We find the following solution:

$$(3)\dots\quad \begin{cases} \rho b_0 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \\ \rho b_1 = a_0 a_1 a_3 - 2a_0 a_2^2 + a_1^2 a_2 \\ \rho b_2 = -a_0 a_2 a_3 + 2a_1^2 a_3 - a_1 a_2^2 \\ \rho b_3 = -a_0 a_3^2 + 3a_1 a_2 a_3 - 2a_2^3 \end{cases}$$

where ρ is an arbitrary factor indicating proportionality merely. We notice immediately that the formulae of transformation are exactly the coefficients of the cubicovariant of the binary cubic

$$a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3,$$

while the twisted cubic curve with which we started is the common intersection of the three quadric surfaces whose equations are

$$(4)\dots a_0 a_2 - a_1^2 = 0,\ a_0 a_3 - a_1 a_3 = 0,\ a_1 a_3 - a_2^2 = 0,$$

whose left-hand members are the coefficients of the Hessian covariant of the binary cubic in question.

The four cubic surfaces of the transformation are tangent to each other along the given twisted cubic, and the discriminant of the binary cubic form put equal to zero:

$$(5)\dots(a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2) = 0$$

is the equation of a quartic developable, whose cuspidal edge is the twisted cubic itself. The Jacobian of the four forms b_0, b_1, b_2, b_3 is moreover the square of this discriminant.

Since the Hessian of the cubicovariant is the product of the discriminant of the Hessian of the given binary cubic, it follows that the three quadric surfaces (4) are converted into themselves by the transformation, as indeed is every quadric surface of the form

$$\lambda(a_0a_2-a_1^2)+\mu(a_0a_3-a_1a_2)+\nu(a_1a_3-a_2^2)=0,$$

while the common intersection of all these surfaces, the given twisted cubic, is converted *point by point* into itself.

To find other special configurations under this transformation, we examine the curves of intersection (other than the twisted cubic common to them all) of the cubic surfaces $b_0=0$, $b_1=0$, $b_2=0$, $b_3=0$ taken two at a time. We find:

$b_0=0$ and $b_1=0$ are tangent to each other along the line $(a_0=0,\ a_1=0)$, which is moreover a double line of the surface $b_0=0$;

$b_2=0$ and $b_3=0$ are, similarly, tangent to each other along the line $(a_2=0,\ a_3=0)$, which is a double line of the surface $b_3=0$;

$b_0=0$ and $b_2=0$ have in common the line $(a_0=0,\ a_1=0)$, which of course is a double intersection, and also the line $(a_1=0,\ a_3=0)$;

$b_1=0$ and $b_3=0$, similarly, have in common as double intersection the line $(a_2=0,\ a_3=0)$ and as simple intersection the line $(a_0=0,\ a_2=0)$;

$b_1=0$ and $b_2=0$ have in common the three lines $(a_0=0,\ a_1=0)$, $(a_1=0,\ a_2=0)$ and $(a_2=0,\ a_3=0)$; and, finally,

$b_0=0$ and $b_3=0$ have in common a second twisted cubic, given parametrically by the relation

$$(6)\ldots.x_0 : x_1 : x_2 : x_3=2\lambda^3 : \lambda^2 : -\lambda : -2$$

It follows immediately (and the results can be verified without difficulty) that: the line $(a_0=0,\ a_1=0)$ is converted into the point $(0:0:0:1)$, and the line $(a_2=0,\ a_3=0)$ into the point $(1:0:0:0)$;

The lines $(a_1=0,\ a_3=0)$ and $(a_0=0,\ a_2=0)$ are interchanged; the line $(a_1=0,\ a_2=0)$ is converted into itself by an involution in which corresponding pairs of points are harmonically conjugate with respect to the double points $(1 : 0 : 0 : 0)$ and $(0 : 0 : 0 : 1)$;

While the line $(a_0=0,\ a_3=0)$ is converted into the twisted cubic (6), and vice versa.

Finally, it is geometrically evident that every point of any tangent to the twisted cubic is transformed into the point of contact of that tangent, and hence that the quartic developable (5) is transformed into the original twisted cubic, its cuspidal edge.

BERKELEY, CALIFORNIA, *December 24, 1901.*

DETERMINATION OF THE GROUP OF RATIONALITY OF A LINEAR DIFFERENTIAL EQUATION.

By DR. SAUL EPSTEEN.

1. When a special algebraic equation is given, its group G can be theoretically determined: 1° by constructing an $n!$-valued function and finding the Galois resolvant; or, 2°, by applying the characteristic double-property of the group, that every rational function of the roots which remains uualtered by all the substitutions of G lies in the domain of rationality; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful:* "If a rational function $\psi(x_1 \ldots . x_n)$ remains formally unaltered by the substitution of a group G' and by no other substitutions, and if ψ equals a quantity lying in the domain of rationality R, and if the conjugates of ψ under $G_{n!}$ are all distinct, then the group of the given equation for the domain R is a subgroup of G'."

A similar situation confronts us when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvant†, or, 2° by applying the Picard-Vessiot characteristic double-property of the group‡, that every rational differential function (of a fundamental system) of the integral which remains unaltered, as a function of x, by all the transformations of G lies in R (the domain of rationality); and conversely. As in the case of algebraic equations both of these methods are impracticable, and in order to find the group a device must be resorted to. With this end in view, I will prove the analog of the above theorem of Algebra, viz: *If a rational differential function (of a fundamental system) of integrals $\psi(y_1 \ldots . y_n, y'_1 \ldots . y'_n, \ldots .)$ remains formally unaltered by the transformations of a complex r-parameter linear homogeneous group G_r; and if, when ψ is transformed by the most general linear homogeneous transformation, it depends on n^2-r essential parameters§, then the group G of the given equation is a subgroup of G_r.*

The advantage of this theorem is that only one function ψ is required, whereas the Picard-Vessiot double theorem relates to the entire infinity of rational functions.

2. Let the given differential equation be

$$(1) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots . + p_n y = 0$$

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* L. E. Dickson, *Theory of Algebraic Equations* (John Wiley and Sons).

† Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, II, part 1, pp. 60 and 68.

‡ Schlesinger, *Handbuch*, II, 1, p. 71.

§That is to say, if G_{n^2} : $\bar{y}_i = \sum\limits_{k=1}^{n} a_{ik} y_k$ is the most general transformation, then $\psi(\bar{y}) = \psi(\Sigma a_{ik} y_k) = \phi(y, A_1 \ldots . A_{n^2-r})$ where the n^2-r A's are all essential. Cf. Picard, *Traité d' Analyse*, III, p. 508; also p. 519.

which is supposed to be special, *i. e.* the coefficients p are known (and not indeterminate) functions of x. A domain of rationality R is supposed to be assigned, this domain containing at least the coefficients. The group of rationality of the differential equation (1) in (R) is in general a complex group made up of ν systems of linear homogeneous transformations

$$G_r : \overline{y}_{ij} = \sum_{k=1}^{n} a_{ikj} y_{kj} \qquad \begin{pmatrix} i=1 \dots n \\ j=0, 1 \dots \nu-1 \end{pmatrix}.$$

$\psi(y)=a(x)$ is *formally* invariant under this r-parameter group. This group G_r contains a certain maximal continuous subgroup Γ which is generated by infinitesimal transformations

$$\Gamma : \overline{y}_{io} = \sum_{k} a_{iko} y_{ko} \qquad (i=1 \dots n).$$

If $T_0=I$, T_1, $\dots T_{\nu-1}$ are the transformations for which

$$T_j^{-1}\, \Gamma\, T_j = \Gamma \qquad (j=0, 1 \dots \nu-1),$$

then G_r is of the form* $G_r : \Gamma, T_1\Gamma, \dots, T_{\nu-1}\Gamma$.

Moreover, $T_o=I$, $T_1 \dots, T_{\nu-1}$ form a substitution-group.

Suppose now that $\phi(y)=r(x)$ is any differential function (as usual, of a fundamental system) of integrals of equation (1), which remains *numerically* unaltered (*i. e.* as function of x) under G_r. It follows that $\phi(y)$ is *formally* unaltered under Γ.† Under the group G_r, ϕ will therefore take ν values: $\phi=\phi_0$, $\phi_1 \dots \phi_{\nu-1}$, and the function

$$\phi = \frac{1}{\nu}(\phi_0 + \phi_1 + \dots + \phi_{\nu-1})$$

remains formally invariant under G_r. It follows now by the Lagrange-Vessiot theorem‡ that ϕ is a rational function of ψ and is therefore a quantity lying in R (*i. e.* $r(x)$ lies in R).

3. We know now that our group G_r has the property that every rational differential function of a fundamental system of integrals $y_1, \dots, y_n$, which remains numerically unaltered by the most general transformation of G_r lies in the domain of rationality R. We will next show that G_r contains as a subgroup (or coincides with) the group of rationality G of the equation (1). Writing

$$(2) \qquad u(y) = A_1 y_1 + \dots + A_n y_n$$

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* Schlesinger, *Handbuch*, II, 1, p. 79.

† Schlesinger, *Handbuch*, II, 1, p. 79, calls θ a characteristic invariant of Γ; cf. also Gino Fano, *Mathematische Annalen*, Vol. 53, p. 403.

‡ Vessiot: *Annales de l' Ecole Normale Superieure*, 1892, p. 223; Schlesinger, II, 1, pp. 53, 57; Picard, *Traite d' Analyse*, III, pp. 521-2.

(the A's being undetermined functions of x) we differentiate n^2 times and eliminate the derivatives of the nth and higher orders by means of the differential equation (1). This gives us n^2+1 equations between the n^2 quantities $y_1 \dots y_n$, $y'_1 \dots y'_n$, $y_1^{(n-1)} \dots y_n^{(n-1)}$; eliminating we obtain a linear differential equation

$$(3) \qquad \frac{d^{n^2}u}{dx^{n^2}} + \pi_1 \frac{d^{n^2-1}u}{dx^{n^2-1}} + \dots + \pi_{n^2}u = 0.$$

We assume that

$$(4) \qquad \theta(u) = 0$$

is a non-linear differential equation of the lowest order with the following properties: its coefficients are rational functions of the A's, their derivatives, and of x; it is irreducible in the sense of Koenigsberger;* at least one of its integrals satisfies (3).†

The equation (2) was differentiated n^2 times; neglecting the results of the last differentiation, we have n^2 equations in the n^2 quantities $y_1 \dots y_n$, $y'_2 \dots y'_n$, $\dots$, $y_1^{(n-1)} \dots y_n^{(n-1)}$. Solving these equations, we obtain

$$y_j^{(i)} = w_{ji1}u + w_{ji2}\frac{du}{dx} + \dots + w_{jin^2}\frac{d^{n^2-1}u}{dx^{n^2-1}}.$$

Substituting the values thus obtained in $\phi(y_1 \dots y_n, \dots, y_1^{(n-1)} \dots y_n^{(n-1)}) = r(x)$ we obtain a rational differential expression

$$(5) \qquad F(u, \frac{du}{dx} \dots \frac{d^\nu u}{dx^\nu}) = r(x) \qquad (\nu \leqq n^2-1).$$

The equation (4) having an integral in common with (5), all of the other integrals of (4) must also satisfy (5).‡ Thus, when we replace u in (5) by any integral of (4), F remains always equal to $r(x)$. The group of rationality of (1) is the totality of transformations which correspond to the passage from some particular integral u_1 of

$$(4) \qquad \theta(u) = 0$$

to all the other integrals of this equation.§ We saw however that $F = r(x)$ for all the integrals of $\theta(u)=0$, hence $\phi(y)$ is numerically invariant under the group of rationality of the equation (1). If

$$\theta(u) \equiv F(u) - r(x) = 0$$

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* Schlesinger, *Handbuch*, II, 1, p. 66.

† *Handbuch*, II, 1, p. 65. In a letter to Picard (*Bulletin des Sciences Mathem.*, March, 1902) Professor Alfred Loewy proved the important theorem that the various equations (4) which are satisfied by the fundamental integrals of (3) are all of the *same* order.

‡ Since (4) is irreducible. Cf. Picard, *Traite*, III, p. 525.

§ *Handbuch*, II, 1, pp. 68-9.

G and G_r will coincide, but in general F may well be of higher order or higher degree than θ and consequently G will be a subgroup of G_r.

4. In the enunciation of the theorem I added that when $\psi(y)$ was transformed by the most general linear homogeneous transformation (in n variables and n^2 parameters) it would become a function of n^2-r essential parameters. This corresponds to the analogous requirement in Algebra that the conjugates of $\psi(x_1 \ldots . x_n)$ under the total symmetric group shall be distinct. The reason for this condition is that it enters in the proof of the Lagrange-Vessiot theorem which was employed in 2.

5. Example 1°: Consider the equation

$$(6) \qquad \frac{d^2y}{dx^2}+y=0.$$

A fundamental pair of integrals is

$$y_1=\sin x, \ y_2=\cos x;$$

between these exists the relation

$$(7) \qquad \psi(y)\equiv y_1^2+y_2^2=1.$$

Now $\psi(y)$ remains formally unaltered under the group G_1:

$$\begin{cases} \bar{y}_1=y_1\cos\alpha-y_2\sin\alpha, \\ \bar{y}_2=y_1\sin\alpha+y_2\cos\alpha; \end{cases} \qquad \begin{cases} \bar{y}_1=y_1\cos\alpha+y_2\sin\alpha, \\ \bar{y}_2=-y_1\sin\alpha+y_2\cos\alpha. \end{cases}$$

When ψ is transformed by

$$\bar{y}_1=a_{11}y_1+a_{12}y_2, \ \bar{y}_2=a_{21}y_1+a_{22}y_2,$$

there results $\psi(\bar{y})\equiv(a_{11}y_1+a_{12}y_2)^2+(a_{21}y_1+a_{22}y_2)^2$, a function of *three* (n^2-r) essential parameters, namely, $A_1y_1^2+A_2y_2^2+A_3y_1y_2$, where $A_1=a_{11}^2+a_{21}^2$, $A_2=a_{12}^2+a_{22}^2$, $A_3=2(a_{11}a_{12}+a_{21}a_{22})$. The group of rationality of (6) is therefore either G_1 or the identity. In the domain* of the trigonometric functions (or their equivalents, such as the exponential functions) the group of rationality of (1) is the identity. In a domain not including the trigonometric functions, G_1 is the required group.

Example 2°: The question of numerical invariance of the rational functions of the integrals was first raised by Klein.† Previous to this Picard‡ had given:

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* For differential equations the domain of rationality is composed of the four fundamental operations, extraction of roots, differentiation and any functions which may be decided on arbitrarily. The functions are however so chosen as to always make the coefficients of the given equation rational.

† F. Klein, *Hohere Geometrie*, II, pp. 298-9. (B. G. Teubner, Leipzig).

‡ Picard, *Comptes Rendus*, 1883, p. 1134. Picard writes the second equation $Y_2=\sqrt{(1-\lambda^2)}y_1-\lambda y_2$; this cannot be correct since $\lambda=1$ gives $Y_2=-y_2$ instead of the identical transformation $Y_2=y_2$.

$$Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2$$
$$Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2$$

as the group of the equation

$$(8) \qquad x(1-x)\frac{d^2y}{dx^2}-\frac{x}{2}\frac{dy}{dx}+a^2y=0, \qquad (a=\text{constant})$$

since the relation between a suitably selected pair of fundamental integrals is $y_1^2+y_2^2=1$.*

On the basis of our theorem however, G_r becomes G_1:

$$\begin{cases} Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2, \\ Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2, \end{cases} \qquad \begin{cases} \bar{Y}_1=\lambda y_1-\sqrt{(1-\lambda^2)}y_2, \\ \bar{Y}_2=\sqrt{(1-\lambda^2)}y_1+\lambda y_2; \end{cases}$$

and the group of rationality of (8) will be G_1 or the identity according to the domain of rationality which is selected. Since (8) is a special case of the hypogeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+[Y-x(1+\alpha+\beta)]\frac{dy}{dx}-\alpha\beta y=0$$

the adjunction of the hypogeometric function would reduce the group of rationality to the identical transformation.

THE UNIVERSITY OF CHICAGO, *December, 1902.*

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* Picard, *loc. cit.*

SOME FALLACIES IN TEXT-BOOKS ON ELEMENTARY SOLID GEOMETRY.

By PROF. G. W. GREENWOOD, McKendree College, Lebanon, Ill.

So much has been written on the fallacies encountered in elementary plane geometries in applying to curves theorems on the lengths of broken lines, that it is, perhaps, superfluous to note the more frequent errors of similar character in elementary texts on solid geometry.

For example, in the case of a cylindrical surface, there is in its definition no connection between it and a prismatic surface; and any attempt to *prove* that the surface of the inscribed or circumscribed prism approaches the surface of the cylinder as the number of sides is indefinitely increased, seems fallacious. For no matter how great is the number of sides of the inscribed prism, there is still an *infinite gap* between it and the cylinder in which it is inscribed, and to assume

that a relation between these surfaces, which initially is *entirely absent*, is supplied by multiplying the number of sides of the prism, is to deceive ourselves.

Equally vulnerable seems the corollary commonly attached, that the lateral area of the cylinder is therefore the limit of the lateral area of the inscribed prism when the number of sides is indefinitely increased; for such a corollary presupposes that we have already defined the *area* of a *curved surface* in some other way.

Such unjustifiable assumptions may be avoided by first showing that the area of the inscribed or circumscribed prism has a limit when the number of sides is indefinitely increased, and then *defining* the area of the cylinder as the limit of the area of the inscribed or circumscribed cylinder.

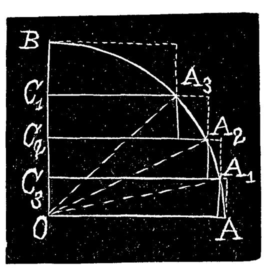
THE VOLUME OF THE SPHERE.

By HENRY L. COAR, A. M., University of Illinois.

The solids of revolution, that is, the solids generated by the revolution of a plane figure about an axis, offer many interesting problems to the student of synthetic geometry of three dimensions. The question of obtaining the volumes and total areas of solids, generated by the revolution of rectilinear figures, presents no particular difficulty, but when we attempt to find by purely synthetic means the volumes and superficial area of solids formed by curvilinear figures, the problem is not always simple without making some assumptions regarding limits. A well-known problem of this kind is to find the volume and area of an anchor-ring. A most interesting problem in this line is that of obtaining the volume of a sphere regarded as a solid of revolution. In the following proof, which I have not been able to find published anywhere, no assumptions of any kind regarding the existence of the limits are necessary.

We need consider only the hemisphere, which is generated by the revolution, through 360°, of a quadrant of a circle about a radius.

Let AOB be the quadrant of a circle and let us divide the radius OB into n equal parts. Then construct a set of inscribed and a set of circumscribed rectangles as indicated in the figure.



If now we rotate the complete figure through 360° about the radius OB, the quadrant will generate a hemisphere, while each of the inscribed as well as each of the circumscribed rectangles will generate a right circular cylinder. Let us designate the sum of the volumes of the cylinders generated by the inscribed rectangles by V_1, that of the cylinders generated by the circumscribed rectangles by V_2, and the volume of the hemisphere by V. We will first prove that both V_1 and V_2 approach V as their limit as n increases indefinitely, and

will then find an expression for either of these and obtain its limiting value. To prove the first we have always $V_1 < V < V_2$, hence

$$V_2 - V < V_2 - V_1 \text{ and } V - V_1 < V_2 - V_1.$$

But $V_2 - V_1$ is the volume of the lowest circumscribed cylinder, *i. e.*

$$V_2 - V_1 = \frac{r}{n} . \pi r^2 = \frac{\pi r^3}{n}$$

where r is the radius of the circle. Hence

$$V_2 - V < \frac{\pi r^3}{n} \text{ and } V - V_1 < \frac{\pi r^3}{n}.$$

It follows therefore that each of these differences can be made less than any assignable positive quantity, and hence

$$\lim_{n \doteq \infty} V_1 = V \text{ and } \lim_{n \doteq \infty} V_2 = V.$$

We thus prove that our hemisphere is actually the limit of the figures in question.

Let us now obtain an expression, say for V_2, and find its limiting value. We see that V_2 is the sum of n right circular cylinders. Let us find their volumes and add. For convenience denote the volumes of these n cylinders, beginning with the lowest one, by $v_1, v_2, \ldots . v_n$. We shall then have

$$v_1 = \frac{r}{n} . AO^2 . \pi = \frac{\pi r^3}{n},$$

$$v_2 = \frac{r}{n} . A_1 C_1^2 . \pi = \frac{\pi r}{n}\left(r^2 - \frac{r^2}{n^2}\right) = \frac{\pi r^3}{n}\left(1 - \frac{1}{n^2}\right),$$

$$v_3 = \frac{r}{n} . A_2 C_2^2 . \pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{2r}{n}\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{2^2}{n^2}\right),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v_n = \frac{r}{n}(A_{n-1} C_{n-1})^2 . \pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{\overline{n-1}\, r}{n}\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{(n-1)^2}{n^2}\right).$$

Hence

$$V_2 = v_1 + v_2 + \ldots . + v_n = \frac{\pi r^3}{n}\left[1 + \left(1 - \frac{1}{n^2}\right) + \left(1 - \frac{2^2}{n^2}\right) + \left(1 - \frac{3^2}{n^2}\right) + \ldots .\right.$$

$$+\left(1-\frac{(n-1)^2}{n^2}\right)\Bigg]$$

$$=\frac{\pi r^3}{n}\left[n-\left(\frac{1}{n}\right)^2-\left(\frac{2}{n}\right)^2-\left(\frac{3}{n}\right)^2-\ldots-\left(\frac{n-1}{n}\right)^2\right]$$

$$=\pi r^3\{1-\frac{1}{n^3}[1^2+2^2+3^2+\ldots+(n-1)^3]\}.$$

Now the series $1^2+2^2+3^2+\ldots+(n-1)^2$ is the sum of the squares of the first $n-1$ positive integers. The formula for the sum of the squares of the first n integers is given in any College Algebra under "piles of shot" and is

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)\,(2n+1)}{6}.$$

Hence

$$1^2+2^2+3^2+\ldots+(n-1)^2=\frac{(n-1)n\,(2n-1)}{6}.$$

Substituting this in the expression for V_2 we have

$$V_2=\pi r^3\left[1-\frac{1}{n^3}.\frac{(n-1)n\,(2n-1)}{6}\right]$$

$$V_2=\pi r^3\left[1-\tfrac{1}{6}(1-\frac{1}{n}).1.(2-\frac{1}{n})\right].$$

Now proceed to the limit and we have

$$V=\lim_{n\doteq\infty}V_2=\pi r^3[1-\tfrac{1}{6}(1).1.(2)]=\pi r^3[1-\tfrac{1}{3}]=\tfrac{2}{3}\pi r^3$$

which is the well-known result.

DERIVATION OF FORMULA FOR Tan $\frac{1}{2}A$ IN SPHERICAL TRIGONOMETRY.

By GEORGE R. DEAN, School of Mines, Rolla, Mo.

Applying Napier's Rules of Circular Parts to the triangle formed by the bisector of angle A, the radius of the inscribed circle and one of the sides, we see that $\sin(s-a)=\tan r\cot\frac{1}{2}A$. Hence, $\tan\frac{1}{2}A=\tan r/\sin(s-a)$, so that

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=\frac{\sin(s-b)}{\sin(s-a)}\ldots.(1).$$

The sine proportion gives

$$\frac{\mathrm{Tan}\frac{1}{2}(A+B)}{\mathrm{Tan}\frac{1}{2}(A-B)}=\frac{\mathrm{Tan}\frac{1}{2}(a+b)}{\mathrm{Tan}\frac{1}{2}(a-b)}\ldots.(2).$$

For convenience, put $\mathrm{Tan}\frac{1}{2}A=x$, $\mathrm{Tan}\frac{1}{2}B=y$, $\mathrm{Tan}\frac{1}{2}(a+b)=m$, $\mathrm{Tan}\frac{1}{2}(a-b)=n$, $\sin(s-b)=p$, $\sin(s-a)=q$. Then

$$\frac{\dfrac{x+y}{1-xy}}{\dfrac{x-y}{1+xy}}=\frac{m}{n}\ldots.(3),$$

$$\text{and }\frac{x}{y}=\frac{p}{q}\ldots.(4).$$

From (3), $\dfrac{1+xy}{1-xy}=\dfrac{x-y}{x+y}.\dfrac{m}{n}$. From (4), $\dfrac{x-y}{x+y}=\dfrac{p-q}{p+q}$.

Therefore, $\dfrac{1+xy}{1-xy}=\dfrac{p-q}{p+q}.\dfrac{m}{n}=\dfrac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}\ldots.(5).$

$$xy=\frac{\tan\frac{1}{2}(a+b)-\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)+\tan\frac{1}{2}c}=\frac{\sin(s-c)}{\sin s}\ldots.(6).$$

Multiplying (4) by (6),

$$x^2=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}=\tan^2\tfrac{1}{2}A.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

CRITICISM ON SOLUTION OF PROBLEM 163, DECEMBER NUMBER.

BY J. M. ARNOLD, CROMPTON, R. I.

The published solution is correct as far as the algebraical part is concerned but when it comes to pairing the couples there is quite a mixing up of the names.

Hendricks is Anna's husband is correct, but Klaus is not Katrine's husband as stated, neither is Hans the husband of Gertrude.

The problem reduces to this: To find three square numbers, which if 63 be added to each the sum in each case will be square. These numbers are readily found by trial to be 1, 81 and 961. We can then make out the following scheme:

	Shillings	Hogs	Hogs	Shillings	
Hans	64	8	1	1	Katrine
Klaus	144	12	9	81	Gertrude
Hendricks	1024	32	31	961	Anna

We see that the third man bought 23 hogs more than the second woman. Hence Hendricks was the name of the third man and Gertrude the name of the second woman. Also the second man bought 11 more than the first woman. Hence, Klaus was the second man and Katrine the first woman. By writing in these names and giving the remaining spaces to the other two names, we shall have each man's name opposite to that of his wife. From which we see that Klaus is Gertrude's husband and not Katrine's as stated.

Also solved by *LON C. WALKER*. Professor Walker also solved No. 164.

ALGEBRA.

164. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

By Sylvester's dialytic method form the eliminant between $mx^3+py^2=0$ ....(1), and $px^2+my^3=0$....(2). Also between $mx^4+py=0$....(1), and $px^3+my^3=0$....(2).

Solution by DR. L. E. DICKSON, The University of Chicago.

The statement of this problem is unfortunate. There is no condition on the coefficients m and p in order that the either pair of equations (1) and (2) shall be simultaneous. The trivial cases in which either m or p is zero will be excluded.

(1) Solution of the simultaneous equations

$$mx^3+py^2=0,\ my^3+px^2=0.$$

Since m and p are not both zero, the determinant

$$\begin{vmatrix} x^3 & y^2 \\ y^3 & x^2 \end{vmatrix}=0, \text{ whence } x^5=y^5.$$

Let $x=\omega y$, where ω is an arbitrary fifth root of unity. Then

$$m\omega^3y^3+py^2\equiv y^2(m\omega^3y+p)=0.$$

Hence there are six sets of solutions:

$$x=0,\ y=0;\ x=-\frac{p}{m}\omega^3,\ y=-\frac{p}{m}\omega^2.$$

(2) Solution of the simultaneous equations

$$mx^4+py=0,\ my^3+px^3=0.$$

In addition to the solutions $x=y=0$, there are exactly nine sets of solutions

$$x=\varepsilon p^{4/9}m^{(-4)/9},\ y=-\varepsilon^4 p^{7/9}m^{(-7)/9},$$

where ε is an arbitrary ninth root of unity.

165. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

Solve $x^4-x=14$, by quadratics.

Solution by G. B. M. ZERR, A.M.. Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$x^4-x=14$ or $x^4-16=x-2$.

$$\therefore x^2-4=\frac{x-2}{x^2+4}=\frac{x}{x^2+4}-\frac{2}{x^2+4}.$$

$$\therefore x^2-\frac{x}{x^2+4}+\frac{1}{4(x^2+4)^2}=4-\frac{2}{x^2+4}+\frac{1}{4(x^2+4)^2}.$$

$$\therefore x=2 \text{ or } x=-2+\frac{1}{x^2+4}.$$

$$\therefore x=2 \text{ or } x^3+2x^2+4x+7=0.$$

$$\therefore x=2 \text{ or } =-1\tfrac{13}{15} \text{ nearly, or } -\tfrac{1}{15}[1\pm\sqrt{(-843)}] \text{ nearly.}$$

166. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve
$$ax+by=2zx\ldots.(1).$$
$$cy+dz=2xy\ldots.(2).$$
$$ez+fx=2yz\ldots.(3).$$

Solution by the PROPOSER.

From (1), (2), and (3), respectively,

$$y=\frac{(2z-a)x}{b}=\frac{dz}{2x-c}=\frac{ez+fx}{2z},$$

whence
$$x(2x-c)(2z-a)=bdz\ldots.(4),$$
$$fx(2x-c)+ez(2x-c)=2dz^2\ldots.(5).$$

From (4), $z=\dfrac{ax(2x-c)}{2x(2x-c)-bd}$, which substituted in (5), gives, after reduction,

$$x^4+\frac{1}{2f}(ae-2cf)x^3+\frac{1}{4f}[fc^2-a^2d-2(ace+bdf)]x^2$$

$$+\frac{1}{8f}[ac(ad+ce)+bd(2cf-ae)]x+\frac{1}{16f}bd(ace+bdf)=0.$$

Similarly,

$$y^4+\frac{1}{2b}(ac-2be)y^3+\frac{1}{4b}[fc^2-e^2b-2(ace+bdf)]y^2$$

$$+\frac{1}{8b}[ce(ae+cf)+df(2be-ac)]y+\frac{1}{16b}df(ace+bdf)=0.$$

$$z^4+\frac{1}{2d}(ce-2ad)z^3+\frac{1}{4d}[a^2d-e^2b-2(ace+bdf)]z^2$$

$$+\frac{1}{8d}[ac(ae+bc)+bf(2ad-ce)]z+\frac{1}{16d}bf(ace+bdf)=0.$$

Also solved by *LON C. WALKER.*

GEOMETRY.

REMARKS ON NO. 187, GEOMETRY, BY J. R. HITT, GOSS, MISS.

There seems to be an error in (4) of Professor Zerr's demonstration of No. 187, Geometry. The result given is not correct. For $t=\frac{1}{\sqrt{2}}b\sin C\cos C$, $t_1=\frac{1}{\sqrt{2}}b\cos C$, $t_2=\frac{1}{\sqrt{2}}b\sin C$, from which it is seen that in general t^2 cannot equal t_1t_2. It is also easily seen that if $t_1 : t=t : t_2$, then must $DI=b$, whereas DI cannot be $>\frac{1}{2}b$.

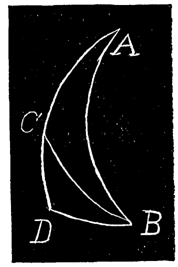
CALCULUS.

154. Proposed by B. R. DOWNER, Hopkinsville, Ky.

At the equinox, when the sun is on the celestial equator, a man starts driving on a perfectly level plain at six o'clock in the morning, and continues, going always from the sun, at the uniform rate of six miles per hour, until six o'clock in the evening. Required the path he will travel and the distance in a straight line from starting point to stopping point.

Solution by the PROPOSER.

Let the line from west to east through the starting point be taken for the axis of X, and that from south to north through the same point for the axis of Y. Let δ be the latitude of the place; then will the inclination of the celestial equator to the vertical circle passing through the east and west points be equal to δ.



Let AB be an arc of this vertical circle extending from the zenith A to the east point B, 90°. Let BC be an arc of the celestial equator, inclined at the angle δ to AB, and AC the vertical circle passing through the sun C at any particular time. If then, z represents the arc of the curve traveled by the man in any given time, BC, the sun's apparent path, expressed in terms of z, will be, for the radius 1, $\pi z/72$. Produce AC until AD is 90°, and draw BD. Then will the angles ABD and D be right angles and the arc BD will measure the angle A, the sun's angular distance at any time from the east point. Then we have from the right angled spherical triangle BCD,

$$\tan A = \sin\delta \tan\frac{\pi z}{72}.$$

But evidently the angle A is the supplement of that which the tangent at any point of the required curve makes with the axis of X.

$$\therefore \frac{dy}{dx} = -\sin\delta\tan\frac{\pi z}{72}. \quad \text{Whence } z = \frac{72}{\pi}\tan^{-1}\left(-\frac{dy/dx}{\sin\delta}\right).$$

$$\therefore dz = -\frac{72}{\pi}\frac{\sin\delta\, d^2y}{dx(\sin^2\delta+(dy^2/dx^2))}. \quad \text{But } dz = -\sqrt{(dx^2+dy^2)}.*$$

$$\therefore \sqrt{1+\frac{dy^2}{dx^2}} = \frac{72}{\pi}\,\frac{\sin\delta\,\frac{d^2y}{dx^2}}{\sin^2\delta+\frac{dy^2}{dx^2}}.$$

$$\text{Let } \frac{dy}{dx} = p. \quad \text{Then } \sqrt{(1+p^2)} = \frac{72}{\pi}\,\frac{\sin\delta(dp/dx)}{\sin^2\delta+p^2}.$$

$$\therefore dx = \frac{72}{\pi}\sin\delta\,\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$$

$$\text{and } pdx = dy = \frac{72}{\pi}\sin\delta\,\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}.$$

* The negative sign of the radical is taken because of the reverse direction in which we have measured the arc z.

Hence we have $x = \frac{72}{\pi}\sin\delta\int\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$.

Put $\sqrt{(1+p^2)}=r-p$. Then $p=\frac{r^2-1}{2r}$, $dp=\frac{r^2+1}{2r^2}dr$, $\sqrt{(1+p^2)}=\frac{r^2+1}{2r}$,

$$\sin^2\delta+p^2=\frac{r^4-2r^2\cos 2\delta+1}{4r^2}.$$

Whence $\int\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}=\int\frac{4rdr}{r^4-2r^2\cos 2\delta+1}=\int\frac{4rdr}{(r^2\cos 2\delta)^2+\sin^2 2\delta}$

Put $r^2-\cos 2\delta=s$. Then $2rdr=ds$, and the integral becomes

$$\int\frac{2ds}{s^2+\sin^2 2\delta}=\frac{2}{\sin 2\delta}\int\frac{ds/\sin 2\delta}{1+[s/(\sin 2\delta)]^2}=\frac{2}{\sin 2\delta}\tan^{-1}\frac{s}{\sin 2\delta}+c.$$

$$\therefore\ x=\frac{72}{\pi\cos\delta}\tan^{-1}\left[\frac{p^2+p\sqrt{(1+p^2)}+\sin^2\delta}{\sin\delta\cos\delta}\right]+c.$$

But from the conditions of the problem, when $x=0$, $p=0$.

$$\therefore\ c=-\frac{72\delta}{\pi\cos\delta},\text{ and }x=\frac{72}{\pi\cos\delta}\tan^{-1}\left[\frac{p^2+p\sqrt{(1+p^2)}+\sin^2\delta}{\sin\delta\cos\delta}\right]-\frac{72\delta}{\pi\cos\delta}.$$

Again we have $y=\frac{72}{\pi}\sin\delta\int\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$.

Put $\sqrt{(1+p^2)}=r$. Then $pdp=rdr$. $\sin^2\delta+p^2=r^2-\cos^2\delta$.

$$\int\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}=\int\frac{dr}{r^2-\cos^2\delta}=\frac{1}{2\cos\delta}\int\frac{dr}{r-\cos\delta}-\frac{1}{2\cos\delta}\int\frac{dr}{r+\cos\delta}$$

$$=\frac{1}{2\cos\delta}\log\left[\frac{r-\cos\delta}{r+\cos\delta}\right]+c'.$$

$$\therefore\ y=\frac{36\tan\delta}{\pi}\log\left[\frac{\sqrt{(1+p^2)}-\cos\delta}{\sqrt{(1+p^2)}+\cos\delta}\right]+c'.$$

But when $y=0$, $p=0$.

$$\therefore c'=-\frac{36\tan\delta}{\pi}\log\tan^2\tfrac{1}{2}\delta.$$

$$\therefore\ y=\frac{36\tan\delta}{\pi}\log\left[\frac{\sqrt{(1+p^2)}-\cos\delta}{\sqrt{(1+p^2)}+\cos\delta}\right]-\frac{36\tan\delta}{\pi}\log\tan^2\tfrac{1}{2}\delta.$$

From the two equations for x and y, we have

$$p^2+p\sqrt{(1+p^2)}=\sin\delta\cos\delta\tan\left[\frac{\pi x\cos\delta}{72}+\delta\right]-\sin^2\delta\ldots.(\text{i}).$$

$$\sqrt{(1+p^2)}=\cos\delta\frac{\cot^2\frac{1}{2}\delta+e^{\pi y/36\tan\delta}}{\cot^2\frac{1}{2}\delta-e^{\pi y/36\tan\delta}}\ldots.(\text{ii}).$$

Denote the second members of equations (i) and (ii) by X and Y. Then we have $p^2+p\sqrt{(1+p^2)}=X$, $\sqrt{(1+p^2)}=Y$. Whence, by eliminating p, we have

$$Y^2=\frac{(X+1)^2}{2X+1}\ldots.(\text{iii}).$$

which is an equation expressing the general character of the curve, which is transcendental. Making $x=0$, we may find two values of y corresponding, which are evidently the starting and stopping points in the problem proposed.

When $x=0$, $X=0$. Hence from (iii) we find $Y=\pm1$, or

$$\cos\delta\frac{\cot^2\frac{1}{2}\delta+e^{\pi y/86\tan\delta}}{\cot^2\frac{1}{2}d-e^{\pi y/36\tan\delta}}=\pm1;$$

whence $e^{\pi y/36\tan\delta}=1$ or $\cot^4\frac{1}{2}\delta$.

$\therefore\ y=0$, or $\dfrac{144\tan\delta}{\pi}\log\cot\frac{1}{2}\delta$, and the distance from starting point to stopping point is the latter value of y.

For the latitude 36° 20′ we have for the distance from starting point to stopping point 37.56 miles.

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\dfrac{2\pi y}{a}\right)+\sinh\left(\dfrac{2\pi y}{a}\right)$.

Solution by the PROPOSER.

Measuring the axis of x perpendicularly to the plane face into the interior of the solid, the longitudinal stresses P, Q and the shearing stress T are functions of x and y only, and the general equations of internal equilibrium reduce to (Thompson and Tait, *Natural Philosophy*, Art. 697) the following:

$$\left.\begin{array}{l} dP/dx+dT/dy=0 \\ dT/dx+dQ/dy=0 \end{array}\right\} \ldots.(1)$$

Let α, β be the displacements of the point originally at (x, y), k=the elasticity of volume, and n=the elasticity of figure of the substance, and let $m=k+\frac{1}{3}n$, then

$$\left.\begin{array}{l} P=(m+n)d\alpha/dx+(m-n)d\beta/dy \\ Q=(m-n)d\alpha/dx+(m+n)d\beta/dy \\ T=n(d\beta/dx+d\alpha/dy) \end{array}\right\} \ldots.(2).$$

Let $\delta=d\alpha/dx+d\beta/dy$=the superficial dilation, and $\nabla^2=d^2/dx^2+d^2/dy^2$, then equations (1) become

$$\left.\begin{array}{l} md\delta/dx+n\nabla^2\alpha=0 \\ md\delta/dy+n\nabla^2\beta=0 \end{array}\right\} \ldots.(3).$$

Assume $P=\varphi(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]$

$Q=\psi(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]$

$T=\theta(x)\left[\cos\left(\frac{2\pi y}{a}\right)+\cosh\left(\frac{2\pi y}{a}\right)\right].$

From equations (1),

$$\varphi'(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]-\frac{2\pi}{a}\theta(x)\left[\sin\left(\frac{2\pi y}{a}\right)-\sinh\left(\frac{2\pi y}{a}\right)\right]=0.$$

$$\theta'(x)+\frac{2\pi}{a}\psi(x)=0.$$

$$\therefore\ \theta(x)=\frac{a}{2\pi}\varphi'(x)\left[\frac{\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)}{\sin\left(\frac{2\pi y}{a}\right)-\sinh\left(\frac{2\pi y}{a}\right)}\right]=\frac{a}{2\pi}B\varphi'(x), \text{ suppose.}$$

$$\therefore\ \theta'(x)=\frac{a}{2\pi}\varphi''(x)B.$$

$$\therefore\ \psi(x)[\sin(2\pi y/a)-\sinh(2\pi y/a)]+(a^2/4\pi^2)\varphi''(x)[\sin(2\pi y/a)$$

$$+\sinh(2\pi y/a)]=0, \text{ or } \psi(x)+B(a^2/4\pi^2)\varphi''(x)=0.$$

From (2), $d\alpha/dx=(P+Q)/4m+(P-Q)/4n$,
$d\beta/dy=(P+Q)/4m-(P-Q)/4n$,

$$\delta=\frac{1}{2m}(P+Q)=\frac{1}{2m}[\varphi(x)+\psi(x)][\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

If $P=pF(y)$ represent the distribution of traction and pressure on the plane face, then $\delta=P/m=(p/m)F(y)$, when $x=0$, and from equations (3). $\nabla^2\delta=0$; δ must also vanish when $x=\infty$.

$$\therefore\ \nabla^2\delta=(1/2m)\{[\varphi''(x)+\psi''(x)][\sin(2\pi y/a)+\sinh(2\pi y/a)]$$
$$-(4\pi^2/a^2)[\varphi(x)+\psi(x)][\sin(2\pi y/a)-\sinh(2\pi y/a)]\}=0.$$

$$\therefore\ B[\varphi''(x)+\psi''(x)]-(4\pi^2/a^2)[\varphi(x)+\psi(x)]=0.$$

$$\therefore B[\varphi''(x)-B(a^2/4\pi^2)\varphi''''(x)]-(4\pi^2/a^2)[\varphi(x)-B(a^2/4\pi^2)\varphi''(x)]=0.$$

$$\therefore\ 2B\varphi''(x)-(4\pi^2/a^2)\varphi(x)-B^2(a^2/4\pi^2)\varphi''''(x)=0.$$

Let $\varphi(x)+\psi(x)=\varphi(x)-B(a^2/4\pi^2)\varphi''(x)=f(x)$.

$\therefore\ Bf''(x)-(4\pi^2/a^2)f(x)=0.\quad \therefore f(x)=Ce^{-(2\pi x/a\sqrt{B})}$.

Let $2\pi/a\sqrt{B}=b.\quad\therefore\ \varphi''(x)-b^2\varphi(x)+b^2Ce^{-bx}=0$.

The solution of this equation is $\varphi(x)=De^{-bx}-\frac{1}{2}bCxe^{-bx}$.

$$\therefore\ \theta(x)=-\frac{abB}{2\pi}De^{-bx}-\frac{abB}{4\pi}Ce^{-bx}+\frac{ab^2B}{4\pi}Cxe^{-bx}.$$

$$\psi(x)=-De^{-bx}-Ce^{-bx}+\tfrac{1}{2}bCxe^{-bx}.$$

Now when $x=0$, $P=Q=p[\sin(2\pi y/a)+\sinh(2\pi y/a)]$. $T=0$; therefore $\varphi(0)=\psi(0)=p$, $\theta(0)=0$; $\therefore$ $D=p$ and $C+2D=0$ or $C=-2p$.

$$\therefore\ P=pe^{-bx}(1+bx)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$Q=pe^{-bx}(1-bx)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$T=-\frac{2\pi dx}{a}e^{-bx}[\cos(2\pi y/a)+\cosh(2\pi y/a)].$$
$$\delta=(p/m)e^{-bx}[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

$$d\alpha/dx=\tfrac{1}{2}pe^{-bx}(1/m+bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$d\beta/dy=\tfrac{1}{2}pe^{-bx}(1/m-bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$\alpha=-(p/2b)e^{-bx}[1/m+1/n(1+bx)][\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$\beta=\tfrac{1}{2}p\int e^{-bx}(1/m-bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

In this last, $b=\dfrac{2\pi}{a}\sqrt{\dfrac{\sin(2\pi y/a)-\sinh(2\pi y/a)}{\sin(2\pi y/a)+\sinh(2\pi y/a)}}$.

DIOPHANTINE ANALYSIS.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for x, y, and z such that the relation $x^2y+xz^2=y^2z$ is satisfied.

II. Solution by W. F. KING, Ottawa, Canada.

We may assume that x, y, and z have no factor common to them all. For if they have a common factor n, each term of the equation $x^2y+xz^2=y^2z$ may be divided by n^3, and there will then be left an equation of the same form, in which x, y, and z have no common factor.

Let the greatest common measure of x and y be a; of y and z, b; and of z and x, c. Then we shall have

$$x=cax_1,$$
$$y=aby_1,$$
$$z=bcz_1.$$

Now observe that since b is the greatest common measure of y and z, ay_1 is prime to cz_1; similarly is bz_1 to ax_1, and cx_1 to by_1.

Hence a, b, and c are all prime to one another; so are x_1, y_1, and z_1 to one another. Also a is prime to z_1, b to x_1, and c to y_1. Substituting in the equation the above values of x, y, and z, and dividing by abc, we have

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1.$$

Divide by x_1; then $a^2c\,x_1y_1+c^2b\,z_1^2=\dfrac{b^2a\,y_1^2z_1}{x_1}$.

The left hand side is integral, therefore $b^2a\,y_1^2z_1$ is divisible by x_1. But as above shown, x_1 is prime to b, y_1 and z_1. Hence a/x_1 must be an integer.

Again, divide the above equation by a. Then

$$acx_1^2y_1+\frac{c^2bz_1^2x_1}{a}=b^2y_1^2z_1.$$

Hence, as before, $\dfrac{c^2b\,z_1^2x_1}{a}$ is an integer. But a is prime to b, c, and z_1. Therefore, x_1/a is an integer. Now since a/x_1 and x_1/a are both integers, $x_1=a$. Similarly, $y_1=b$, $z_1=c$. Therefore the equation

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1$$

becomes $a^4bc+abc^4=ab^4c$, or dividing by abc, $a^3+c^3=b^3$, a known impossible form.

102. Proposed by F. L. SAWYER. Mitchell, Ontario, Canada.

Prove that the factors of the sum of the squares of two numbers prime to each other are themselves the sum of two squares.

Solution by DR. L. E. DICKSON, The University of Chicago.

If a and b are relatively prime, every prime divisor of a^2+b^2 is of the form $4n+1$; inversely, every prime of the form $4n+1$ can be expressed as the sum of the squares of the two relatively prime integers. These are well known theorems in the Theory of Numbers (compare Weber's *Algebra*, 1st edition, I, p. 585). Let a^2+b^2 have the prime factors $p_1, p_2, \ldots, p_s$, which need not be distinct. In view of the theorems quoted, $p_i = x_i^2 + y_i^2$, where x_i and y_i are integers. But

$$(x_i^2+y_i^2)(x_j^2+y_j^2)=(x_ix_j+y_iy_j)^2+(x_iy_j-x_jy_i)^2.$$

It fellows that any prime or composite divisor of a^2+b^2 is the sum of two squares.

Also solved by *LON C. WALKER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

165. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A borrows \$2000 and agrees to pay back principal and interest in 100 equal monthly payments. Find the monthly payment. What would he have to pay yearly on the same conditions in order to discharge the debt in 100 months?

ALGEBRA.

173. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $\sqrt{(a+x+y)}=z\ldots.(1)$, $\sqrt{(b+y+z)}=x\ldots.(2)$, $\sqrt{(c+z+x)}=y\ldots.(3)$.

GEOMETRY.

195. Proposed by F. L. SAWYER, Mitchell. Ontario, Canada.

The diagonals of a four-sided figure are h and k, and the area is A; show that the area of the circumscribing square is

$$\frac{h^2k^2-4A^2}{h^2+k^2-A}.$$

196. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a quadrilateral circumscribe a circle, the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

CALCULUS.

161. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

A cylindrical oil tank of length l and radius r is capped by curved ends and rests with the axis horizontal. The total axial length of tank is $l+2h$. If the oil stands at depth d in the tank (d less than $2r$) find its volume (a) when the ends are portions of the surface of a sphere, (b) when the ends are portions of the surface of an ellipsoid.

A special case of the above problem was recently received from Houston, Texas, for solution. It is passed on to the readers of the MONTHLY.

162. Proposed by J. E. SANDERS, Hackney, O.

Solve the differential equations

$$(a)\ x\frac{dy}{dx}-y=x\sqrt{(x^2+y^2)},\quad (b)\ \cos x\frac{dy}{dx}+y=1-\sin x.$$

MECHANICS.

151. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

An elastic ball is projected along a horizontal tube, striking first the bottom, then the top, then the bottom, and so on. Find the number of times the ball will strike the top.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A ball of lead an inch in diameter is cast into a ball of rubber two inches in diameter so as to be internally tangent. What is the nature of the path made by this *lead-rubber* ball in rolling down an inclined plane?

DIOPHANTINE ANALYSIS.

112. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the nth triangle.

AVERAGE AND PROBABILITY.

136. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

By direct computation find the average distance between two points in the surface of a given rectangle, but on opposite sides of a diagonal.

137. Proposed by G. H. HARVILL, Malakoff, Texas.

A, B, C, and D, in playing whist, agree that the person who first cuts an ace shall have a stake of \$313. What is the value of each person's expectation before the play begins, each taking his turn at cutting in the order named as the game progresses?

NOTES.

Professor Luigi Cremona, of Rome, has been elected a foreign member of the American Academy of Arts and Sciences, of Boston. F.

Any of our readers knowing of a vacancy in mathematics, or of a prospective vacancy, will confer a favor on us by informing us of the same. We are in a position to recommend a number of most excellent candidates for such vacancies. F.

On December 30th, 1902, editor, Dr. L. E. Dickson, was married to Miss Susan M. Davis, of Waco, Texas. After the wedding, the fortunate couple started on an extended trip through Old Mexico, remaining several weeks at the City of Mexico. Congratulations are now in order. F.

BOOKS AND PERIODICALS.

The Elements of Plane and Spherical Trigonometry. By T. U. Taylor, C. E., University of Texas, and C. Puryear, M. A., C. E., Agricultural and Mechanical College of Texas. Large 8vo. Cloth. 160 pages of text, 67 pages of tables. Price, $1.25. 1902. Boston: Ginn & Co.

The authors have succeeded in realizing their aim to present the essential parts of trigonometry in a form adapted to practical applications. The chief novelty lies in the excellent sets of new exercises and problems of a practical character. Some disappointment is felt over the omission of the projective proofs of the addition formulae. On page 46 functions are indicated with the angle omitted. Napier's rules are not merely an aid to the memory, but of added value in that they enable us to pick out without device that one of the ten formulae needed in a given case. While noting the occurance of the vowels *a* and *o* in Napier's Rules, no mention is made of the vowel *i* in *sine* and *middle*. The book appears to possess pedagogical excellence, and the typography is attractive. D.

Solid Geometry. Revised Edition. By G. A. Wentworth. 8vo. xvi+218 pages. Price $0.75. 1902. Boston: Ginn & Co.

The book is a reprint of pages 251-459 of Wentworth's revised Plane and Solid Geometry (1899), with modified table of formulas and index, and ten introductory pages giving the necessary references to plane geometry. D.

An Elementary Treatise on the Mechanics of Machinery with special reference to the Mechanics of the Steam Engine. By Joseph N. Le Conte, Instructor in Mechanical Engineering, University of California, Associate member of the American Institute of Electrical Engineers, etc. 8vo. Cloth, x+311 pages. Price, $2.25. New York: The Macmillan Co.

The book is divided into three parts. The first two embody the more important principles of what may be called the Kinematics of Machinery, and the third part treats of the Mechanics of the Steam Engine, kinematically and dynamically. The work presents in a most excellent manner the applications of the principles of mechanics to certain prob-

lems connected with machinery. All the problems are clearly worked out and illustrated with diagrams neatly constructed. This book will prove of great value both to the theoretical mechanic and the practical machinist. F.

Elementary Applied Mechanics. By T. Alexander, C. E., M. Inst. C. E. I., M. A. I. (Hon. Causa), 4th Order of Meiji, Japan, Professor of Engineering, Trinity College, Dublin, and A. W. Thomson, D. Sc., Professor of Engineering, College of Science, Poona. With numerous diagrams, and a series of graduated examples carefully worked out. 8vo. Cloth, xx+575 pages. Price, $5.25. New York: The Macmillan Co.

In this work the following subjects are treated very fully: Internal stress and strain; transverse stress; bending moments, and shearing forces for fixed loads, for combined fixed loads, and for moving loads; resistance, in general, to bending and shearing at the various cross-sections of framed girders and solid beams; stress at an internal point of a beam; curvature, slope, and deflection; and a number of other allied subjects. Each subject treated is fully illustrated with the solutions of a number of problems bearing on it. The work forms an elementary consecutive course on the subject of internal stress and strain, based on the late Professor Rankine's treatment of the subject in his *Applied Mechanics* and *Civil Engineering.* It is the best elementary treatise that has yet appeared. F.

Manual of Advanced Optics. By C. Riborg Mann, Assistant Professor of Physics in the University of Chicago. 8vo. Cloth, 196 pages. Price, $2.00. Chicago: Scott, Forseman & Co.

The object of this work is to meet the needs of the more advanced students of Optics. While there are many excellent works treating the theory of optics very exhaustively, yet none treat of some of the recent and important discoveries. No work thus far, to my knowledge, treats of the practical applications of that marvelous little instrument, the Interferometer, invented by Dr. Michelson. In the work before us is found not only the treatment of the adjustment and use of the interferometer, but also the use of defracting and concave gratings, the prism, spectrometer, etc. Dr. Mann has the happy faculty of clearing up the difficulties of the subject, and this book will be greatly appreciated by all students interested in the subject of Optics. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published quarterly, under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance.

No. 1 of Vol. XXV contains the following articles: The Parametric Representation of the Tetrahedroid Surface, by D. M. Lehmer; On Ternary Monomial Substitution-Groups of Finite Order with Determinant + or —1, by E. B. Skinner; On the Forms of Unicusal Sextic Scrolls, and On the Forms of Sextic Scrolls of Genus One, by Virgil Snyder; Note on Symmetric Functions, by E. D. Roe, Jr. F.

The Annals of Mathematics. Edited by Ormond Stone, W. E. Byerly, and others. Published Quarterly under the auspices of Harvard University. Price, $2.00 per year in advance.

No. 2 of Vol. 4, second series, contains the following articles: The Logarithm as a Direct Function, by J. W. Bradshaw, with an Introduction by W. F. Osgood; On Positive Quadratic Forms, by Paul Saurel; Multiple Points on Lissajous's Curves in Two and Three Dimensions, by Edward A. Hook; A Special Quadri-Quadric Transformation of Real Points in a Plane, by C. C. Engberg. F.

# THE AMERICAN MATHEMATICAL MONTHLY.

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AN ELEMENTARY EXAMPLE OF MODULAR SYSTEMS.

By PROF. G. A. MILLER.

The following exposition is based upon Kronecker's *Lectures on Number Theory*, pp. 144-150.* While the example is very elementary and does not lead to any new results, yet it may serve to give some idea of the general modular systems, and also to exhibit a fundamental theorem of arithmetic in an attractive manner.

The number† a is said to be divisible by m whenever it is possible to find a number c such that $a=cm$. Similarly we may say that a is divisible by the system of numbers $m_1, m_2, \dots, m_\alpha$ provided it is possible to find α numbers $c_1, c_2, \dots, c_\alpha$ such that

$$a=c_1m_1+c_2m_2+\dots+c_\alpha m_\alpha .$$

From this standpoint the system of numbers $m_1, m_2, \dots, m_\alpha$ is called a *modular system* and it is denoted by $(m_1, m_2, \dots, m_\alpha)$. The numbers $m_1, m_2, \dots, m_\alpha$ are called the *elements* of the system. For instance, 3 is divisible by the modular system (7, 16, 25) because $3=3.7+2.16-2.25$. In particular, zero is divisible by every modular system since

$$0=0.m_1+0.m_2+\dots+0.m_\alpha .$$

Vorlesungen ueber Zahlentheorie von Leopold Kronecker, Erster Band, Leipzig, 1901, B. G. Teubner.
†Only integers, including 0, are meant by the term *number* as used in this note.

One modular system $(m_1, m_2, \dots, m_\alpha)$ is said to be divisible by another $(d_1, d_2, \dots, d_\beta)$ whenever each element of the first system is divisible by the second system; *i. e.*, when the following α equations are satisfied:

$$m_1 = c_{11}d_1 + c_{12}d_2 + \dots + c_{1\beta}d_\beta ,$$
$$m_2 = c_{21}d_1 + c_{22}d_2 + \dots + c_{2\beta}d_\beta ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$m_\alpha = c_{\alpha 1}d_1 + c_{\alpha 2}d_2 + \dots + c_{\alpha\beta}d_\beta .$$

For instance, the system (20, 35) is divisible by the system (15, 25, 10) since

$$20 = 0.15 + 0.25 + 2.10,$$
$$35 = 1.15 + 0.25 + 2.10.$$

From the following equations it follows that the system (15, 25, 10) is also divisible by the system (20, 35):

$$15 = 1.35 - 1.20,$$
$$25 = 3.35 - 4.20,$$
$$10 = 2.35 - 3.20.$$

Two modular systems which have the property that each of them is divisible by the other are called *equivalent*. The above examples prove that the two systems (20, 35) and (15, 25, 10) are equivalent. This relation may be indicated by the symbol (20, 35) $\backsim$ (15, 25, 10). It may be observed that we are here dealing with an interesting *extension of ordinary division*. If each of two equivalent modular systems is composed of only one element these two elements must be equal as in ordinary division.

If a number is divisible by $(m_1, m_2, \dots, m_\alpha)$ it is evidently also divisible by $(d_1, d_2, \dots, d_\beta)$, since the latter is a divisor of the former. In particular, if a number is divisible by a given modular system it is also divisible by every equivalent system. If each one of two numbers is divisible by $(m_1, m_2, \dots, m_\alpha)$ it is evident that their sum and their difference are divisible by the same system; *i. e.*, the multiples of a given modular system reproduce themselves when they are combined with respect to the two operations, addition and subtraction. Hence they must also reproduce themselves with respect to multiplication.

Since the multiples of two equivalent systems are identical, it is of interest to determine the simplest system which is equivalent to a given system $(m_1, m_2, \dots, m_\alpha)$. It is not difficult to see directly that this consists of a single element, viz. the greatest common divisor of the elements $m_1, m_2, \dots, m_\alpha$. That is, *each modular system of numbers is equivalent to some modular system composed of one element.* If the elements of the modular system are functions of one or more

variables this result is not generally true.* The following considerations furnish a simple proof of the theorem stated above when $m_1, m_2, \ldots, m_\alpha$ are numbers.

Let m be any number which is divisible by $(m_1, m_2, \ldots, m_\alpha)$ and consider the two systems

$$(m, m_1, m_2, \ldots, m_\alpha) \text{ and } (m_1, m_2, \ldots, m_\alpha).$$

Since every element of each of these systems is divisible by the other system we have $(m, m_1, m_2, \ldots, m_\alpha) \sim (m_1, m_2, \ldots, m_\alpha)$.

That is, *if any multiple of the elements of a modular system be added to its elements the resulting system is equivalent to the original system.*

In particular, the two systems $(m_1+tm_2, m_1, m_2, \ldots, m_\alpha)$ and $(m_1, m_2, \ldots, m_\alpha)$ are equivalent for all values of t. The former of these is clearly equivalent to $(m_1+tm_2, m_2, \ldots, m_\alpha)$ since

$$m_1=(m_1+tm_2)-tm_2.$$

Hence it follows that $(m_1, m_2, \ldots, m_\alpha) \sim (m_1+tm_2, m_2, \ldots, m_\alpha)$.

That is, *any modular system is equivalent to the one obtained by increasing or decreasing one of its elements by any multiple of any other element of the system.*

Suppose that a given modular system involves at least two elements which differ from 0. The preceding theorem enables us to find an equivalent system in which one of the elements is reduced by an integer. Hence every modular system is equivalent to a system in which all the elements except one are zeros. Since two systems which differ only with respect to 0 elements are equivalent, it follows that every modular system is equivalent to a system containing only one element, as was stated above. That is, it is always possible to find a number d such that

$$(m_1, m_2, \ldots, m_\alpha) \sim (d).$$

It remains only to find d. Since each of the elements $m_1, m_2, \ldots, m_\alpha$ is divisible by d it follows that d is a common divisor of the elements of $(m_1, m_2, \ldots, m_\alpha)$. The other condition imposed by this equivalence, viz.,

$$d=c_1m_1+c_2m_2+\ldots+c_\alpha m_\alpha,$$

requires that d be divisible by the greatest common divisor of $m_1, m_2, \ldots, m_\alpha$. Hence d must be the greatest common divisor of the elements of

$$(m_1, m_2, \ldots, m_\alpha).$$

We are now in position to see more clearly why the two systems

$$(20, 35) \quad (15, 25, 10)$$

---

*All the modular systems considered in this note are supposed to have numbers for their elements.

are equivalent, as each of them is equivalent to (5).

Suppose that the greatest common divisor of the elements of the system $(m_1, m_2, \ldots, m_a)$ is unity. From

$$(m_1, m_2, \ldots, m_a) \backsim (1)$$

it follows that it is possible to find a numbers $c_1, c_2, \ldots, c_a$ such that

$$c_1 m_1 + c_2 m_2 + \ldots + c_a m_a = 1.$$

In particular, we have the fundamental theorem that it is possible to find two numbers x, y such that

$$m_1 x + m_2 y = 1,$$

whenever m_1 and m_2 are prime to each other.

GENERALIZATION OF A FUNDAMERTAL THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By PROF. M. W. HASKELL.

The theorem in question is of fundamental importance in the geometry of the triangle,* and may be stated as follows:

If A', B', C' be points chosen at will on the sides BC, CA, AB of any triangle ABC, the circles $AB'C'$, $BC'A'$, $CA'B'$ pass through one and the same point O.

In a communication presented to the Chicago Section of the American Mathematical Society January 2, 1902, I extended this theorem to the tetrahedron in the following form:

Let F, G, H, P, Q, R be any points on the edges AD, BD, CD, BC, CA, AB, respectively, of any tetrahedron; the four spheres $AFQR$, $BGRP$, $CHPQ$, $DFGH$ pass through one and the same point O.

The theorem is, however, capable of generalization to space of any number of dimensions without any difficulty. I will therefore state and prove it at once for space of n dimensions,—understanding by a spherical space of three dimensions, S_3, a space every section of which by a flat space of three dimensions, R_3, is an ordinary sphere, and in general by a spherical S_{n-1} a space every section of which by a flat R_{n-1} is a spherical S_{n-2}. A spherical S_{n-1} will evidently be determined by $n+1$ points of which never more than two lie on the same line nor more than three in the same plane, etc. The general theorem may then be stated in the following words:

*McClelland, *Geometry of the Circle*, page 40; see also Rouche et de Comberousse, *Traite de Geometrie*, 7th edition, Vol. I, page 486.

Let A_1, A_2, $A_3 \dots A_{n+1}$ be the vertices of an $(n+1)$-point in a flat space of n dimensions, and select at will on each edge A_iA_k of this $(n+1)$-point a point A_{ik}. The $n+1$ spherical S_{n-1} determined by the groups of points such as $[A_i;\ A_{i1}, A_{i2}, \dots A_{i,\,n+1}]$ will all pass through one and the same point O.

We shall use barycentric coördinates a', a'', $\dots a^{[n+1]}$ with reference to the given n-point, so that

$$\Sigma a^{[i]} = a' + a'' + \dots + a^{[n+1]} = K.$$

These coördinates are the generalization of triangular coördinates in the plane, in which case K is the area of the triangle of reference, and of tetrahedral coördinates in space of three dimensions, where K is the volume of the fundamental tetrahedron. For the problem in hand it is important to regard the absolute values of these coördinates, and not, as in projective geometry, merely their ratios.

The equation of the region at infinity, a flat R_{n-1}, is then

$$\Sigma a^{[i]} = 0\,;$$

and, if we denote by a_{ik} the length of the edge A_iA_k, the equation of the spherical S_{n-1} circumscribing the fundamental $(n+1)$-point $A_1 A_2 \dots A_{n+1}$ will be

$$\Omega = \sum_{i=1}^{n} \sum_{k=i+1}^{n+1} a^2{}_{ik} a^{[i]} a^{[k]} = 0,$$

while the equation of *any* spherical S_{n-1} will be of the form

$$\Sigma \lambda^{[i]} a^{[i]} . \Sigma a^{[i]} - \Omega = 0,$$

where the $\lambda^{[i]}$ are constant coefficients.

Now the coördinates of any vertex A_i of the fundamental n-point are of course all zero except $a_i{}^{[i]}$, which is equal to K; and the coördinates of any of the intermediate points A_{ik} are all zero except two, $a_{ik}{}^{[i]}$ and $a_{ik}{}^{[k]}$, whose sum is equal to K. If for brevity we write

$$\omega_i = \sum_{k=1}^{n+1} a_{ik}{}^2 a_{ik}{}^{[i]} a^{[k]}, \ [k \gtrless i]$$

the equation of the spherical S_{n-1} through the points $[A_i;\ A_{i1}, A_{i2}, \dots A_{i,\,n+1}]$ is readily found to be

$$\omega_i \Sigma a^{[i]} - K\Omega = 0.$$

We see however that

$$K\Omega = \Sigma \omega_i a^{[i]}$$

identically, and hence that *every one of the spherical S_{n-1} in question passes through the point O for which*

$$\omega_1 = \omega_2 = \omega_3 = \ldots = \omega_{n+1},$$

and the theorem is proved.

We proceed to the discussion of some special cases.

I. If the points A_{ik} are the middle points of the edges, it is evident that

$$\omega_i = \frac{K}{2} \sum_{k=1}^{n+1} a_{ik}{}^2 \alpha^{[k]} = \frac{K}{2} \frac{\partial \Omega}{\partial \alpha^{[i]}},$$

and the point O is the center of the spherical S_{n-1} circumscribed about the fundamental n-point.

II. If the flat spaces

$$\omega_1 = 0, \; \omega_2 = 0, \; \ldots . \omega_{n+1} = 0$$

have a point in common, the point O will be that point and it will then lie on the circumscribing S_{n-1}.

Now this will be the case, for example, if, when n is *even*, the points A_{ik} are the intersections with the edges of a flat R_{n-1}. For, let the equation of this R_{n-1} be

$$\lambda_1 \alpha' + \lambda_2 \alpha'' + \ldots + \lambda_{n+1} \alpha^{[n+1]} = 0.$$

The coordinates of A_{ik} will then be

$$\alpha_{ik}{}^{[i]} = \frac{\lambda_k . K}{\lambda_k - \lambda_i}, \quad \alpha_{ik}{}^{[k]} = \frac{\lambda_i . K}{\lambda_i - \lambda_k}$$

and the resultant of the equations $\omega_1 = 0$ will be the determinant

$$\begin{vmatrix}
0 & \frac{a_{12}^2 \lambda_2 K}{\lambda_2 - \lambda_1} & \frac{a_{13}^2 \lambda_3 K}{\lambda_3 - \lambda_1} & \cdots\cdots & \frac{a^2{}_{1,\, n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_1} \\
\frac{a_{21}^2 \lambda_1 K}{\lambda_1 - \lambda_2} & 0 & \frac{a_{23}^2 \lambda_3 K}{\lambda_3 - \lambda_2} & \cdots\cdots & \frac{a^2{}_{2,\, n+1} \lambda_{n+2} K}{\lambda_{n+1} - \lambda_2} \\
\frac{a_{31}^2 \lambda_1 K}{\lambda_1 - \lambda_3} & \frac{a_{32}^2 \lambda_2 K}{\lambda_2 - \lambda_3} & 0 & \cdots\cdots & \frac{a^2{}_{3,\, n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_3} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{a^2{}_{n+1,\, 1} \lambda_1 K}{\lambda_1 - \lambda_{n+1}} & \frac{a^2{}_{n-1,\, 2} \lambda_2 K}{\lambda_2 - \lambda_{n+1}} & \frac{a^2{}_{n-1,\, 3} \lambda_3 K}{\lambda_3 - \lambda_{n+1}} & \cdots\cdots & 0
\end{vmatrix}$$

If we now divide the columns of this determinant respectively by λ_1, λ_2, λ_3, $\ldots\lambda_{n+1}$, the quotient is a zero-axial skew determinant, of *odd* order when n is *even* and therefore vanishing. In this case, then, the flat spaces $\omega_1=0$, $\omega_2=0$, $\omega_3=0$, $\ldots\omega_{n+1}=0$ have a point in common; this will be the point O and it lies on the circumscribing S_{n-1}. In particular, for the case of a plane triangle $(n=2)$, if the points A_{23}, A_{31}, A_{12} are collinear, the point O lies on the circumscribing circle,—a known theorem.

If, however, n is odd, the above determinant does not vanish. In this case the point O is a point of the intersecting R_{n-1}. For, since O is determined by

$$\omega_1=\omega_2=\omega_3=\ldots=\omega_{n+1},$$

we may write

$$\omega_1-\rho=0,\ \omega_2-\rho=0,\ \ldots\omega_{n+1}-\rho=0,$$

and the resultant of these $n+1$ equations and of the equation of the given R_{n-1} is the determinant

$$\begin{vmatrix}
0 & \dfrac{a_{12}^2\lambda_2 K}{\lambda_2-\lambda_1} & \dfrac{a_{13}^2\lambda_3 K}{\lambda_3-\lambda_1} & \cdots\cdots & \dfrac{a^2_{1,\,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_1} & -1 \\
\dfrac{a_{21}^2\lambda_1 K}{\lambda_1-\lambda_2} & 0 & \dfrac{a_{23}^2\lambda_3 K}{\lambda_3-\lambda_2} & \cdots\cdots & \dfrac{a^2_{2,\,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_2} & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\dfrac{a^2_{n+1,\,1}\lambda_1 K}{\lambda_1-\lambda_{n+1}} & \dfrac{a^2_{n+1,\,2}\lambda_2 K}{\lambda_2-\lambda_{n+1}} & \dfrac{a^2_{n+1,\,3}\lambda_3 K}{\lambda_3-\lambda_{n+1}} & \cdots\cdots & 0 & -1 \\
\lambda_1 & \lambda_2 & \lambda_3 & \cdots\cdots & \lambda_{n+1} & 0
\end{vmatrix}$$

and, if we divide the first $n+1$ columns of this determinant by λ_1, λ_2, λ_3, $\ldots\lambda_{n+1}$, respectively, the quotient is a zero-axial skew determinant, which is of odd order if n is *odd*, and will then vanish.

In particular, for the case of a tetrahedron $(n=3)$, if the points A_{ik} are coplanar, the point O lies on the same plane, and we have a correspondence between points and planes, in which to every plane is coördinated a point lying in that plane. The relation is not, however, uniquely reversible, as in the case of the ordinary null-system, for to every point O correspond *six* planes.

Finally, if we consider the S_{n-1} in sets of n, each such set will have a second point of intersection in addition to the point O. These points will be situated in the respective faces of the n-point,—meaning by faces the flat R_{n-1} determined by the vertices taken n at a time. It is evident that, if n is odd, these points will lie on the circumscribing S_{n-1}; while if n is even, they will lie on the the R_{n-1} whose intersections with the edges determine the points A_{ik}.

Rome, Italy, *December 15, 1902.*

INTEGRATION AS A SUMMATION.

By PROF. GEORGE R. DEAN, Rolla, Mo.

The following method of presenting the matter seems worthy of a place in elementary text-books, but I have not yet seen it in print.

Let the integral $(a,\ a+h)$ be divided into n equal parts and find the limit, as n is increased without limit, of the expression

$$f(a).\frac{h}{n}+f(a+\frac{h}{n}).\ \frac{h}{n}+f(a+\frac{2h}{n}).\ \frac{h}{n}+\ldots+f(a+\frac{(n-1)h}{n})\frac{h}{n}+f(a+h)\frac{h}{n}.$$

Supposing $f(x)$ capable of development by Taylor's Formula, we have

$$f(a+\frac{h}{n})=f(a)+f'(a).\frac{h}{n}+\frac{f''(a)}{2!}\frac{h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{h^r}{n^r}+\ldots$$

$$f(a+\frac{2h}{n})=f(a)+f'(a)\frac{2h}{n}+\frac{f''(a)}{2!}\frac{2^2h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{2^rh^r}{n^r}+\ldots$$

. . . .

$$f(a+\frac{rh}{n})=f(a)+f'(a)\frac{rh}{n}+\frac{f''(a)}{2!}\frac{r^2h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{r^rh^r}{n^r}+\ldots$$

Then $\sum\limits_{r=0}^{r=n} f(a+\frac{rh}{n})\frac{h}{n}=hf(a)+\frac{h^2}{n^2}f'(a)\sum\limits_{r=0}^{r=n} r+\frac{h^3}{n^3}f''(a)\sum\limits_{r=0}^{r=n} r^2+\ldots$

$$+\frac{h^{p+1}}{n^{p+1}}f^p(a)\sum_{r=0}^{r=n} r^p+\ldots$$

By Chrystal's *Algebra*, t. I, p. 487, $\sum\limits_{r=0}^{r=n} r^p=\frac{1}{p+1}n^{p+1}+q_1n^p+\ldots.$

$$\underset{n=\infty}{L}\frac{h^{p+1}}{n^{p+1}}\sum_{r=0}^{r=n} r^p=\frac{h^{p+1}}{p+1}.$$

Then $\underset{n=\infty}{L}\sum\limits_{r=0}^{r=n} f(a+\frac{rh}{n})\frac{h}{n}=hf(a)+\frac{h^2}{2!}f'(a)+\frac{h^3}{3!}f''(a)+\ldots$

If $f(a)$ is the derivative of $\varphi(a)$, we have, by Taylor's Theorem,

$$\varphi(a+h)-\varphi(a)=hf(a)+\frac{h^2}{2!}f'(a)+\frac{h^3}{3!}f''(a)+\cdots$$

$$\therefore\ \varphi(a+h)-\varphi(a)=\underset{n=\infty}{L}\sum_{r=0}^{r=n}f(a+\frac{rh}{n})\frac{h}{n}.$$

Hence $\int_a^{a+h}f(x)dx=\varphi(a+h)-\varphi(a)$, where $\varphi(x)$ is the anti-derivative of $f(x)$.

ON THE CHINESE ORIGIN OF THE SYMBOL FOR ZERO.

By PROFESSOR FLORIAN CAJORI.

I have just received a letter from Mr. Y. Mikami, of Tokyo, Japan, containing information which (if confirmed by more extended research) is of great interest and importance. The letter is dated December 15, 1902. From it I quote the following:

"I have found very important relations between the mathematics of India and of China. Arabian numerals seem to be of Chinese origin. The abacus, used by the Chinese from time immemorial, probably afforded the principle of position. In China the use of the symbol 0 for zero seems to have been very old. I desire to study the history of the Chinese mathematics from this point of view, if only I can secure sufficient materials, which is, however, very difficult. Chinese works are not [difficult] to understand for us Japanese, because we use the same letters."

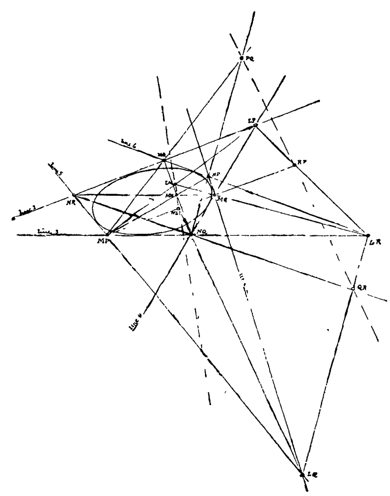
Until recently the symbol for zero and the principle of local value in our notation of numbers were supposed to be of Hindu origin. A few years ago our attention was called to the early work of the Japanese, and now the priority appears to be passing to the Chinese.

COLORADO COLLEGE, COLORADO SPRINGS, *January 3, 1903.*

THE DERIVATION OF THE BRIANCHON CONFIGURATION FROM TWO SPATIAL POINT-TRIADS.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina, Chapel Hill, N. C.

Cayley* has considered the question of deriving the Pascalian configuration, by projection, from a pair of trihedrals. Denote the three planes of one trihedral by a_1, a_2, a_3; of the other by b_1, b_2, b_3. Considering the nine lines $a_i\, b_j \left\{\begin{matrix} i=1, & 2, & 3 \\ j=1, & 2, & 3 \end{matrix}\right\}$ and taking them in a particular way in six sets of three each, we may pass hyperboloids through each set of three lines. These hyperboloids intersect in four points O_1, O_2, O_3, O_4 and if we project the solid figure of the two trihedrals from any one of these four points upon an arbitrary plane, the resulting figure is the Pascalian configuration.



This theorem of Cayley's, in connection with certain considerations concerning cubic surfaces, led to the present investigation. Four-point coördinates are used in this paper and a word of explanation is perhaps not amiss. An equation of the form

$$u_1 x + u_2 y + u_3 z + u_4 w = 0$$

is the equation of a *point*, the coördinates being the variables u_1, u_2, u_3, u_4, which are the perpendiculars from the four points A, B, C, D of the fundamental tetrahedron $ABCD$ upon any plane passing through the point in question. If we have two points given by their equations

$$\xi = 0, \; \xi' = 0,$$

then the equation

$$\xi + \kappa \xi' = 0 \quad (\kappa \text{ constant})$$

represents some point on the line joining the two given points. These preliminary remarks should suffice to make the succeeding discussion readily comprehensible.

Consider two point-triads in space, the one triad consisting of the points designated L, M, N, the other consisting of the points designated P, Q, R. In the system of four-point coördinates chosen above, where u_1, u_2, u_3, u_4 are current coördinates, the equations of the points are chosen as follows:

*Collected Mathematical Papers, Vol. VI, pp. 129-134.

$$
\begin{aligned}
&L : u_4=0\\
&M : mu_1+lu_2+lmnu_3+(mn-l)(nl-m)(lmn-1)u_4=0\\
&N : nlu_1+mnu_2+u_3-(mn-l)(nl-m)(lmn-1)u_4=0
\end{aligned}
$$

and

$$
\begin{aligned}
&P : u_1-(mn-l)(lmn-1)u_4=0\\
&Q : u_2-(nl-m)(lmn-1)u_4=0\\
&R : u_3+(mn-l)(nl-m)u_4=0.
\end{aligned}
$$

Suppose we are given a point, it is possible to find its polar plane with respect to a surface of the third order. If this surface degenerates into three planes (distinct), the problem is still possible. In like manner, if we are given the equations of three points, in four-point coördinates, it is always possible to determine uniquely the pole of a given plane with respect to this system of three points.

Let v_1, v_2, v_3, v_4 denote constant values of u_1, u_2, u_3, u_4, respectively; $\left.\frac{\partial F}{\partial u_1}\right]_{u_1=v_1} \equiv \frac{\partial F}{\partial v_1}$, and similarly in other cases.

The initial problem is to find a plane such that its pole with respect to the system of points, written in the symbolic form

$$LMN=0\ldots.(1)$$

is identical with its pole with respect to the second system of three points, written

$$PQR=0\ldots.(2)$$

The pole

$$\frac{\partial F}{\partial v_1}u_1+\frac{\partial F}{\partial v_2}u_2+\frac{\partial F}{\partial v_3}u_3+\frac{\partial F}{\partial v_4}u_4=0$$

of the plane (v_1, v_2, v_3, v_4) with respect to the system (1) given by the equation

$$F(u_1, u_2, u_3, u_4)=0$$

has for its equation

$$
\begin{aligned}
&[2lmnv_1v_4+n(l^2+m^2)v_2v_4+m(l^2n^2+1)v_3v_4+\lambda\mu^2\nu.v_4^2]u_1\\
&+[2lmnv_2v_4+n(l^2+m^2)v_1v_4+l(m^2+n^2)v_3v_4+\lambda^2\mu\nu.v_4^2]u_2\\
&+[2lmnv_3v_4+m(l^2n^2+1)v_1v_4+l(m^2+n^2)v_2v_4-\lambda\mu\nu^2.v_4^2]u_3\\
&+[lmn(v_1^2+v_2^2+v_3^2)-3\lambda^2\mu^2v^2.v_4^2+n(l^2+m^2)v_1v_2+m(l^2m^2+1)v_1v_3\\
&\qquad+l(m^2+n^2)v_2v_3+2\lambda\mu^2\nu.v_1v_4+2\lambda^2\mu\nu.v_1v_4+2\lambda^2\mu\nu.v_3v_4\\
&\qquad-2\lambda\mu\nu^2.v_3v_4]u_4=0\ldots.(3)
\end{aligned}
$$

where λ, μ, $\nu\equiv mn-l$, $nl-m$, $lmn-1$, respectively. Also the pole

$$\frac{\partial\Phi}{\partial v_1}u_1+\frac{\partial\Phi}{\partial v_2}u_2+\frac{\partial\Phi}{\partial v_3}u_3+\frac{\partial\Phi}{\partial v_4}u_4=0$$

of the plane (v_1, v_2, v_3, v_4) with respect to the system (2) given by the equation

$$\Phi(u_1, u_2, u_3, u_4)=0$$

has for its equation

$$\begin{aligned}&[v_2v_3-\mu\nu.v_3v_4+\lambda\mu.v_2v_4-\lambda\mu^2\nu.v_4^2]u_1\\&+[v_1v_3-\lambda\nu.v_3v_4+\lambda\mu.v_1v_4-\lambda^2\mu\nu.v_4^2]u_2\\&+[v_1v_2-\lambda\nu.v_2v_4-\mu\nu.v_1v_4+\lambda\mu\nu^2.v_4^2]u_3\\&+[-\lambda\nu.v_2v_3-\mu\nu.v_1v_3+2\lambda\mu\nu^2.v_3v_4+\lambda\mu.v_1v_2-2\lambda^2\mu\nu.v_2v_4\\&\quad-2\lambda\mu^2\nu.v_1v_4+3\lambda^2\mu^2\nu^2.v_4^2]u_4=0\ldots.(4)\end{aligned}$$

where λ, μ, $\nu\equiv mn-l$, $nl-m$, $lmn-1$, respectively, as before.

Now it is evident by inspection that equations (3) and (4) are identical (aside from sign) of

$$v_1=v_2=v_3=0.$$

Accordingly the plane of the face ABC of the fundamental tetrahedron $ABCD$ is such that its pole with respect to the point-triad (1) is coincident with its pole with respect to the point-triad (2).

Connect up the six points L, M, N, P, Q, R by lines and planes in every possible way. Suppose the plane of ABC to be intersected by the line LM in the point LM, and by the plane LMN in the line LMN; and so in the other cases. We obtain in this fashion a configuration in the plane of ABC, consisting of the fifteen $(\equiv{}_2C_6)$ points LM, LN, ...QR, and of the twenty $(\equiv{}_3C_6)$ lines LMN, LMP, ...PQR; and which is such that through each of the points there pass four of the lines, and on each of the lines lie three of the points. Thus the lines

$$\left.\begin{matrix}LMN\\LMP\\LMQ\\LMR\end{matrix}\right\}\text{pass through the point } LM;$$

and the points

$$\left.\begin{matrix}LM\\MN\\NL\end{matrix}\right\}\text{lie on the line } LMN;$$

and so in other cases.

It will next be shown that six lines, denoted 1, 2, 3, 4, 5, 6, may be drawn in the plane of ABC, conditioned as follows:

$$(\mathrm{A})\left\{\begin{array}{lllll}\text{line } (1) & \text{passes} & \text{through the} & \text{points} & LP,\ MQ,\ NR \\ \text{“} \ (2) & \text{“} & \text{“} & \text{“} & LQ,\ MR,\ NP \\ \text{“} \ (3) & \text{“} & \text{“} & \text{“} & LR,\ MP,\ NQ \\ \text{“} \ (4) & \text{“} & \text{“} & \text{“} & LP,\ MR,\ NQ \\ \text{“} \ (5) & \text{“} & \text{“} & \text{“} & LQ,\ MP,\ NR \\ \text{“} \ (6) & \text{“} & \text{“} & \text{“} & LR,\ MQ,\ NP \end{array}\right.$$

For this purpose, represent any line in the plane of ABC as the join of two points whose equations are

$$\left.\begin{array}{l}\lambda_1 u_1+\mu_1 u_2+\nu_1 u_3=0\\ \lambda_2 u_1+\mu_2 u_2+\nu_2 u_3=0\end{array}\right\}$$

If this line meets the line LP, the join of the two points, whose equations are

$$\begin{array}{l}L:\ u_4=0\\ P:\ u_1-(mn-l)(lmn-1)u_4=0\end{array}$$

we have the equation of condition

$$\begin{vmatrix}\lambda_1, & \mu_1, & \nu_1, & 0\\ \lambda_2, & \mu_2, & \nu_2, & 0\\ 0, & 0, & 0, & 1\\ 1, & 0, & 0, & -(mn-l)\,(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1:\mu_2=\nu_1:\nu_2,$$

and hence the line in question may be written

$$\left.\begin{array}{c}u_4=0\\ \mu_1 u_2+\nu_1 u_3=0\end{array}\right\}.$$

If this line meets the line MQ also, we have the equation of condition

$$\begin{vmatrix}1, & 0, & 0, & 0\\ 0, & \mu_1, & \nu_1, & 0\\ m, & l, & lmn, & (mn-l)(nl-m)(lmn-1)\\ 0, & 1, & 0, & -(nl-m)(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1:\nu_1=1:l$$

and hence the required line has for its equations

$$1:\begin{bmatrix}u_1=0\\ u_2+lu_3=0\end{bmatrix}.$$

If now we write the equation of the point N in the form

$$N:\ (nlu_1+mnu_2+lmnu_3)-(lmn-1)[u_3+(mn-l)(nl-m)u_4]=0$$

and note the equation of the point R

$$R:\ u_3+(mn-l)(nl-m)u_4=0,$$

it is evident that the equations of the line NR may be written

$$NR:\left\{\begin{matrix} nlu_1+mnu_2+lmnu_3=0 \\ u_3+(mn-l)(nl-m)u_4=0\end{matrix}\right\}$$

That the line 1 meets the line NR is now obvious by inspection.

Determining in similar fashion the equations of the five remaining lines, we obtain

$$2:\left\{\begin{matrix} u_2=0 \\ mu_1+u_3=0\end{matrix}\right\}$$

$$3:\left\{\begin{matrix} u_3=0 \\ nu_1+u_2=0\end{matrix}\right\}$$

$$4:\left\{\begin{matrix} u_1=0 \\ lu_2+u_3=0\end{matrix}\right\}$$

$$5:\left\{\begin{matrix} u_2=0 \\ u_1+mu_3=0\end{matrix}\right\}$$

$$6:\left\{\begin{matrix} u_3=0 \\ u_1+nu_2=0\end{matrix}\right\}$$

Now these six lines 1, 2, 3, 4, 5, 6 touch the conic given by the equation

$$lmn(u_1^2+u_2^2+u_3^2)+mn(l^2+1)u_2u_3+nl(m^2+1)u_3u_1+lm(n^2+1)u_1u_2=0.$$

This is most easily shown by putting u_1, u_2, u_3 in turn equal to zero; we obtain respectively,

$$mn(u_2+lu_3)(lu_2+u_3)=0,$$
$$nl(mu_1+u_3)(u_1+mu_3)=0,$$
$$lm(nu_1+u_2)(u_1+nu_3)=0.$$

Moreover it is clear from an inspection of the scheme (A) above that the points LP, LQ, LR; MP, MQ, MR; NP, NQ, NR are the points 14, 25, 36; 35, 16, 24; 26, 34, 15, respectively, where 14, for example, denotes the meet of the lines 1 and 4; and so in other cases.

Conversely, starting from the six lines 1, 2, 3, 4, 5, 6 touching the above conic, and denoting the points 14, 25, 36; 35, 16, 24; 26, 34, 15 (which are indeed the vertices, and meets of opposite sides of the six-side 162435) in the man-

ner described above, then it is possible to complete the figure of the fifteen points LM, LN, ...QR and of the twenty lines LMN, LMP, ...PQR, such that through each point pass four lines, and on each line lie three points, as detailed in the foregoing.

Of the fifteen points, nine, viz. the points LP, LQ, LR; MP, MQ, MR; NP, NQ, NR are, as appeared above, points on two of the six lines 1, 2, 3, 4, 5, 6; the remaining points are MN, NL, LM; QR, RP, PQ. These are *Brianchon points*

MN	of the six-side	162435
NL	"	152634
LM	"	142536
QR	"	152436
RP	"	142635
PQ	"	162534,

for the point MN is the meet of lines MNP, MNQ, $MNR \equiv MP$, NP; MQ, NQ; MR, $NR \equiv 35$, 26; 16, 34; 24, 15; that is, MN is the Brianchon point of the six-side 162435; and similar reasoning verifies the above statements for the rest of the six-lines.

To summarize, we have two sets of three six-sides such that the Brianchon points of each set lie *in linea;* and the two lines so obtained together with the eighteen lines through the six Brianchon points, form a system of twenty lines passing by fours through fifteen points.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Deduce the Sylvestrian Reciprocant of $ax^3+3bx^2y^2+ay^3+d=0$.

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Differentiating, dividing by 3, and combining, we have

$$a(x^2+y^2\frac{dy}{dx})+2bxy(y+x\frac{dy}{dx})=0....(1).$$

Repeating the operation, we have

$$a\left[x+2y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2}\right]+2b\left[y^2+4xy\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^2+x^2y\frac{d^2y}{dx^2}\right]=0\ldots(2).$$

Eliminating a and b in equations (1) and (2), we have

$$\begin{vmatrix} x^2+y^2\dfrac{dy}{dx}, & xy\left(y+x\dfrac{dy}{dx}\right) \\ x+2y\left(\dfrac{dy}{dx}\right)^2+y^2\dfrac{d^2y}{dx^2}, & \left(y+2x\dfrac{dy}{dx}\right)^2+x^2y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2 \end{vmatrix}=0,$$

which is the Sylvestrian Reciprocant of $ax^3+3bx^2y^2+ay^3+d=0$, since this function would have the same form if x were the dependent and y the independent variable.

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A weight of m pounds falls and is broken into n pieces after which it is found that all weights, in pounds, from 1 to m can be weighed. Find the weight of each piece. Apply when $m=121$, $n=5$.

I. Solution by the PROPOSER.

Let x_1, x_2, x_3, ...x_n be the n pieces. Then $x_1+x_2+x_3+\ldots.+x_{n-1}+x_m=m$, $x_2-x_1=x_1+1$, or $x_2=2x_1+1$.

$x_3-x_2-x_1=x_2+x_1+1$, or $x_3=2x_2+2x_1+1=3(2x_1+1)=3x_2$.

$x_4-x_3-x_2-x_1=x_3+x_2+x_1+1$, or $x_4=2x_3+3x_2=9(2x_1+1)=9x_2$.

Generally, $x_r=3^{r-2}(2x_1+1)=3^{r-2}x_2$.

$\therefore x_1+x_2+x_3+\ldots.+x_n=x_1+(2x_1+1)(1+3+9+27+\ldots.+3^{n-2})=m$.

$\therefore x_1+(2x_1+1)(3^{n-1}-1)=2m$ or $x_1\dfrac{2m+1-3^{n-1}}{2.3^{n-1}}$.

$x_2=(2x_1+1)=\dfrac{2m+1}{3^{n-1}}$; $x_r=3^{r-2}x_2=\dfrac{2m+1}{3^{n-r+1}}$.

When $m=121$, and $n=5$, x_1-1, $x_2=3$, $x_3=9$, $x_4=27$, $x_5=81$.

II. Solution by FRANK L. GRIFFIN, Graduate Student, The University of Chicago.

Let $f(n)$=number of groupings of n weights in two groups; then the maximum number giving one group a preponderence is $f(n)/2$.

Now, $f(n)=3^n-1$....(i) [for proof see below]. Hence, to weigh all weights, in pounds, from 1 to m by using n weights, it is necessary that

$$m\overline{\lesseqgtr}\frac{3^n-1}{2}\ldots.(\text{ii}).$$

By using the n weights, 1, 3, 9, ...3^{n-1}, all weights, in pounds, from 1 to

their sum, $S_n\left[=\frac{3^n-1}{2}\right]$ can be obtained; *i. e.* from 1 to [or beyond] m....(iii). [For proof of (iii) also see below].

Therefore, we arrive at the following solution:

A. When $m=\frac{3^n-1}{2}=S_n$.

The n weights, 1, 3, 9, 27, ...3^{n-1}, satisfy both requirements: (a) sum of weights$=m$, (b) possibility of obtaining all weights from 1 to m.

B. When $m<\frac{3^n-1}{2}$.

The (n—1) weights, 1, 3, 9, ...3^{n-2}, together with $(m-S_{n-1})$ as the other, satisfy requirements (a) and (b).

C. When $m>\frac{3^n-1}{2}$ there is no solution.

In the particular case where $m=121$, $n=5$, $m=\frac{3^n-1}{2}$, therefore the weights are 1, 3, 9, 27, and 81. [If m had been between 41 and 121, n would still have to be as much as 5, but the last weight would have been less than 81, (or$=m-40$)].

PROOF OF (i). Let $(A)....(B)$ be any grouping obtainable with n weights; then by using one more weight, w_{n+1}, we get groupings as follows:

$$(A)....(B);\ (A+w_{n+1})....(B);\ (A)....(B+w_{n+1}).$$

Or for $(n+1)$ weights there are $3.f(n)$ groupings and in addition the two:

$$(w_{n+1})....(0) \text{ and } (0)....(w_{n+1}).$$

$\therefore f(n+1)=3.f(n)+2....(1)$.

Assume $f(n)=3^n-1$; then from (1), $f(n+1)=3^{n+1}-1$. Whence by induction, the formula holds for all values of n as it is evidently true for $n=1$, $n=2$.

PROOF OF (iii). Assume that with k weights, 1, 3, 9, ...3^{k-1}, all weights up to $S_k\left(=\frac{3^k-1}{2}\right)$ can be obtained. Then, since $w_{k+1}=3^k=2S_k+1$,

$$S_k+c=w_{k+1}-(s_k+1-c).$$

Therefore, any weight S_k+c between S_k and S_{k+1} can be obtained by combining with w_{k+1} a grouping (S_k+1-c), which is not less than S_k and hence is obtainable by weights, w_1, w_2, ...w_k. Therefore by using the first $(k+1)$ weights, all weights up to S_{k+1} can be obtained.

Hence, by induction, since by weights w_1, w_2, all weights up to $s_2(=4)$

can be obtained, all weights from 1 to $\frac{3^n-1}{2}$ can be obtained by using the weights $1, 3, 9, 27, \ldots 3^{n-1}$.

Also solved by *J. SCHEFFER*, and *J. E. SANDERS*.
No solution of problem 168 has yet been received.

GEOMETRY.

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178. Proposed by JOHN M. ARNOLD, Crompton, R. I.

A cylinder thirty feet long and two feet in diameter is to be placed in a machinery car, the inside dimensions of which are eight feet wide and eight feet high. Find length of shortest car that will contain it.

Solution by the PROPOSER.

Consider the car standing on one end, which we shall call the base. Fig. 2.

If a projection of the cylinder be made on the base, each end will be projected into an ellipse with its minor axis on the diagonal MN of the base.

Fig. 1 is a vertical plane taken on the line MN. [In Fig. 2, the points H, D, E, and F correspond to the same points in Fig. 1.]

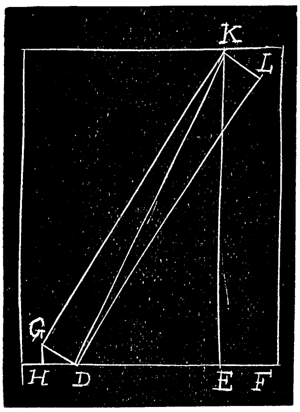


Fig. 1.

As the sides of the square in Fig. 2 are tangents to the equal ellipses, we have by Analytical Geometry $MO=\sqrt{(A^2+B^2)}$, where A and B are the semi-major and semi-minor axes,

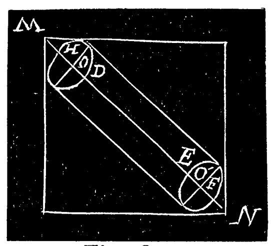


Fig. 2.

$$2\overline{MO}+\overline{OO'}=\overline{MN}=8\sqrt{2}\ldots.(1).$$

In the similar triangles GDH and DLF, $GD : HD=DL : LF$, or $2A : 2B=30 : \frac{30B}{A}$. Then $DF=OO'=\sqrt{900-\frac{900B^2}{A^2}}$.

Substituting values of OO' and MO in (1),

$$2\sqrt{(A^2+B^2)}+\sqrt{900-\frac{900B^2}{A^2}}=8\sqrt{2}.$$

A=radius of cylinder=1. Substituting and reducing

$$12769B^4-21728B^2=-9184,$$

from which $B^2=.918904$ and $B=.9586$. Substituting the value of B^2 in the expression for DF, $DF=OO'=8.5432$. Then $DE=OO'-2B=6.626$, and $EK=\sqrt{[(30)^2+(2)^2-(6.626)^2]}=29.3274$ feet, or 29 feet 3.9288 inches, the length of car required.

Also solved by the late *P. H. PHILBRICK*, who obtained as a result, 29.168 feet.

CALCULUS.

155. Proposed by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Solve the differential equations:

$$(A).\quad \frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}=\sin 2x+\sin x-x. \quad (B).\quad \frac{d^2y}{dx^2}+2\frac{dy}{dx}=\sin 2x+\sin x-x.$$

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, O., and LON C. WALKER, A. M., Leland Stanford University.

Using the symbolic method (A) becomes $(D^4+2D^2)y=\sin 2x+\sin x-x$. The complementary function is $c_1+c_2x+c_3\cos\sqrt{2}x+c_4\sin\sqrt{2}x$, and the particular integral

$$=\frac{1}{D^2(D^2+2)}(\sin 2x+\sin x-x)=\frac{1}{D^2}\cdot\frac{1}{D^2+2}\sin 2x+\frac{1}{D^2}\cdot\frac{1}{D^2+2}\sin x-$$

$$\frac{1}{D^2+2}\cdot\frac{1}{D^2}x=\tfrac{1}{8}\sin 2x-\sin x-(2+D^2)^{-1}.\frac{x^3}{6}=\tfrac{1}{8}\sin 2x-\sin x-\frac{x^3}{12}+\tfrac{1}{4}x.$$

$\therefore\ y=c_1+c_2x+c_3\cos\sqrt{2}x+c_4\sin\sqrt{2}x+\frac{1}{8}\sin 2x-\sin x-\frac{1}{12}x^3$ ($\frac{1}{4}x$ being included in c_2x); and (B) becomes $(D^2+2D)y=\sin 2x+\sin x-x$.

$\therefore$ The complementary function is $c_1+c_2e^{-2x}$, and the particular integral

$$=\frac{1}{D^2+2D}(\sin 2x+\sin x-x)=\frac{1}{D^2+2D}\sin 2x+\frac{1}{D^2+2D}\sin x-\frac{1}{D+2}\cdot\frac{1}{D}x$$

$$=\frac{1}{2D-4}\sin 2x+\frac{1}{2D-1}\sin x-(2+D)^{-1}.\tfrac{1}{2}x^2$$

$$=\tfrac{1}{2}.\frac{D+2}{D^2-4}\sin 2x+\frac{2D+1}{4D^2-1}\sin x-(\tfrac{1}{2}-\tfrac{1}{4}D+\tfrac{1}{8}D^2)\tfrac{1}{2}x^2$$

$$=-\tfrac{1}{16}(D+2)\sin 2x-\tfrac{1}{5}(2D+1)\sin x-\tfrac{1}{4}x^2+\tfrac{1}{4}x-\tfrac{1}{8}$$

$$=-\tfrac{1}{8}\cos 2x-\tfrac{1}{8}\sin 2x-\tfrac{2}{5}\cos x-\tfrac{1}{5}\sin x-\tfrac{1}{4}x^2+\tfrac{1}{4}x-\tfrac{1}{8}.$$

$\therefore\ y=c_1+c_2e^{-2x}-\frac{1}{8}(\cos 2x+\sin 2x)-\frac{2}{5}\cos x-\frac{1}{5}\sin x-\frac{1}{4}x^2+\frac{1}{4}x$, ($-\frac{1}{8}$ being included in the term c_1).

Also solved by *J. SCHEFFER*, *W. W. LANDIS*, *G. W. GREENWOOD*, and *WILLIAM HOOVER*.

156. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume common to the two solids $x^2+y^2+z^2=a^2$ and $xz^2=(a-x)(x^2+y^2)$.

I. Solution by the PROPOSER.

The limits of z are $z=\sqrt{\frac{a-x}{x}(x^2+y^2)}$ to $z=\sqrt{(a^2-x^2-y^2)}$.

Eliminating z, $y=\sqrt{[x(a-x)]}$.

$\therefore$ The limits of y are 0 and $\sqrt{[x(a-x)]}=y'$; the limits of x are 0 and a.

$$\therefore\ V=4\int_0^a\int_0^{y'}\left[\sqrt{(a^2-x^2-y^2)}-\sqrt{\frac{a-x}{x}(x^2+y^2)}\right]dxdy,$$

$$=2\int_0^a\left[(a^2-x^2)\sin^{-1}\sqrt{\frac{x}{a+x}}-x\sqrt{(ax-x^2)}\log\left(\frac{\sqrt{a}+\sqrt{(a-x)}}{\sqrt{x}}\right)\right]dx.$$

Let $x=a\tan^2\theta$ in the first term, and $x=a\sin^2\theta$ in the second term.

$$\therefore\ V=4a^3\int_0^{\frac{1}{4}\pi}\theta(1-\tan^4\theta)\tan\theta\sec^2\theta d\theta-4a^3\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^2\theta\log\left(\frac{1+\cos\theta}{\sin\theta}\right)d\theta$$

$$=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3+\tfrac{1}{4}a^3\int_0^{\frac{1}{2}\pi}\log(\tan\tfrac{1}{2}\theta)d\theta.$$

$$\int_0^{\frac{1}{2}\pi}\log(\tan\tfrac{1}{2}\theta)d\theta=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\log\left(\frac{1-\cos\theta}{1+\cos\theta}\right)d\theta=-\int_0^{\frac{1}{2}\pi}(\cos\theta+\tfrac{1}{3}\cos^3\theta+\tfrac{1}{5}\cos^5\theta+\dots\dots)d\theta$$

$$=-2\int_0^{\frac{1}{2}\pi}(\cos\theta+\tfrac{1}{3}\cos3\theta+\tfrac{1}{5}\cos5\theta+\dots.)d\theta=-2(1-1/3^2+1/5^2-1/7^2+1/9^2-\dots)$$

$$=-16\left[\frac{1}{1^2.3^2}+\frac{3}{5^2.7^2}+\dots+\frac{2n-1}{(4n-3)^2(4n-1)^2}\right]$$

$=-1.832$ nearly$=-16[.114488335]$.

$\therefore\ V=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3-.458a^3=\tfrac{2}{3}\pi a^3-1.708a^3$.

II. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Transforming the two surfaces to polar coördinates, the equations of the first are $\rho=a$, and of the second, $\rho=a\sin\phi\sec\theta$. The equations of the intersection of the two surfaces are $\left.\begin{matrix}\rho=a\\ \sin\phi=\cos\theta\end{matrix}\right\}$.

The volume common to the two solids bounded by the surfaces is

$$V=4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{3}\pi-\phi}\int_{a\sin\phi\sec\theta}^{\frac{1}{2}\pi}\rho^2\sin\phi d\phi d\theta d\rho=\tfrac{4}{3}a^3\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}(1-\sin^3\phi\sec^3\theta)\sin\phi d\phi d\theta,$$

$$=\tfrac{4}{3}a^3[\tfrac{1}{2}\pi-\tfrac{15}{16}+\tfrac{3}{16}\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3+\tfrac{1}{4}a^3\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi],$$

$$=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3-\tfrac{1}{4}a^3\sum_{n=1}^{n=\infty}\frac{2n-1}{[4n-3]^2\,[4n-1]^2}=.386a^3,\text{ nearly.}$$

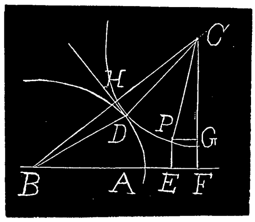
On the integration of $\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi$, see the remarks on Prize Problem, No. 123, Calculus.

157. Proposed by **L. C. WALKER, A. M.,** Graduate Student, Leland Stanford Jr. University, Cal.

Two equal ellipses are tangent to each other at the vertices of the major axes. If one of them be rolled on the other; find (1) the equation and area of the curve described by the vertex, and (2) by the center.

Solution by **G. B. M. ZERR, A.M., Ph.D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the **PROPOSER.**

Let a and b be the semi-axes of both ellipses; B the center of the fixed ellipse: C, the center of the rolling ellipse; P, its vertex; and D, the point of contact; $e^2=(a^2-b^2)/a^2$. Let $BE=x$, $BF=m$, $PE=y$, $CF=n$, and $\angle ABC=\angle PCB=\theta$.



Then, $BC=2a\sqrt{(1-e^2\sin^2\theta)}$=twice the length of the perpendicular from B on the tangent at D.

$m=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}$, $n=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}$,

$x=m-PG=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}-a\cos2\theta$,

$y=n-CG=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}-a\sin2\theta$.

$x^2+y^2=r^2=a^2+4a^2(1-e^2\sin^2\theta)-4a^2\cos\theta\sqrt{(1-e^2\sin^2\theta)}$,

the equation of the locus of the vertex.

$$\text{Area}=a^2\int_0^{\pi}[5-4e^2\sin^2\theta-4\cos\theta\sqrt{(1-e^2\sin^2\theta)}]d\theta=\pi a^2(5-2e^2)=\pi(3a^2+2b^2).$$

$BC^2=\rho^2=4a^2(1-e^2\sin^2\theta)$, the equation of the locus of the center.

$$\text{Area}=4a^2\int_0^{\pi}(1-e^2\sin^2\theta)d\theta=2\pi a^2(2-e^2)=2\pi(a^2+b^2).$$

Also solved by *J. SCHEFFER*, and *G. W. GREENWOOD*.

MECHANICS.

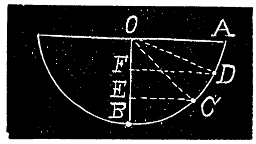
147. Proposed by **W. J. GREENSTREET, M. A.,** Editor of The Mathematical Gazette, Stroud, England.

A particle mass m is attached to one end of a string, the other end of which is fixed. It is projected horizontally with such a velocity that it would rise to a position in which

the string would be horizontal. But on its upward path it meets an inelastic particle mass m' and the height to which it rises is diminished by $1/p$th of what it would have risen. Find m', and the tensions of the string just after collision and at the greatest height of the particle.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let B be the point of projection of particle mass m; C, the position of m' supposed at rest; D, the point at which m stops after impact with m'. Let $\angle COB=\theta$, $BO=a$. Then $OF=a/p$, $FB=a(p-1)/p$, $OE=a\cos\theta$. Velocity of projection $=\sqrt{(2ga)}$, velocity of m at C at moment of impact $=\sqrt{(2ga\cos\theta)}$.



$\therefore m\sqrt{(2ga\cos\theta)}=v'(m+m')$.

But $v'=\sqrt{(2g.FE)}=\sqrt{[(2ga/p)(p\cos\theta-1)]}$.

$\therefore m\sqrt{(2ga\cos\theta)}=(m+m')\sqrt{[(2ga/p)(p\cos\theta-1)]}$.

$$\therefore m'=\frac{m\{\sqrt{[\cos\theta]}-\sqrt{[(p\cos\theta-1)/p]}\}}{\sqrt{[(p\cos\theta-1)/p]}}.$$

Let $(m+m')g=W$. The tension is composed of the tension due to acceleration imparted by the central force plus the tension due to gravity. Let $T=$ tension just after impact, $t=$tension at highest point.

Then $T=\dfrac{Wv'^2}{ga}+W\cos\theta=(2W/p)(p\cos\theta-1)+W\cos\theta=3W\cos\theta-2W/p$.

Since velocity is zero at D, $t=W\cos DOB=W/p$. $\therefore t=W/p$.

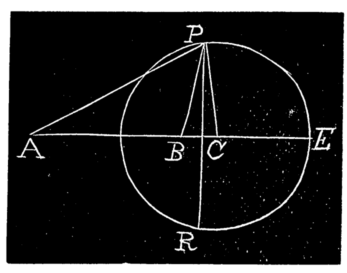
As m' is inelastic, it is supposed that m' coalesces with m.

148. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centers of gravity should lie on the surface of the sphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the electrical images, a=radius of sphere, C its center, $AC=x$, $BC=y$. Let e=charge of electricity situated at A; and let the sphere be insulated and have a charge $-(ea^3/x^3)$. Let PR be the line of no electrification. Let m=mass of each point on surface of sphere; u=mass of point at A; v=mass of point at B; $\angle BAP=\theta$. Now $AP=AC$, $BP=PC$ (since PR is line of no electrification). Also



$u+v=2m....(1)$,

$u(x-a)=v(a-y)....(2)$.

From triangles ACP and ABP we have, since $AP=AC=x$, $BP=PC=a$,

$$\cos\theta=\frac{2x^2-a^2}{2x^2}=\frac{x^2+(x-y)^2-a^2}{2x(x-y)}.$$

$\therefore x=a^2/y$. Let $y=a/n$. $\therefore x=an=n^2y$. From (1) and (2),

$$u=\frac{2m}{1+n},\quad v=\frac{2mn}{1+n},\quad u=\frac{v}{n}.$$

Hence, the law: The distance, from the center of the sphere of the point without varies as the square of the distance of the point within from the center of the sphere. The mass of the point without varies directly as the mass of the point within and inversely as the distance from the center.

DIOPHANTINE ANALYSIS.

103. Proposed by HARRY S. VANDIVER, Bala, Pa.

Find some solutions of $x^3+ay^3=z^3$ (for x, y, and z) and show that there is an infinite number of solutions corresponding to each integral value of a.

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

For generality we may write

$$x^3+mx^2y+nxy^2+ay^3=x^2 \ldots .(1).$$

Let us consider the following products.

$$(u+\alpha v+\alpha^2w)(u+\beta v+\beta^2w)(u+\gamma v+\gamma^2w) \ldots .(2), \text{ and}$$
$$(u'+\alpha v'+\alpha^2w')(u'+\beta v'+\beta^2w')(u'+\gamma v'+\gamma^2w') \ldots .(3),$$

where α, β, γ are the roots of $\theta^3-m\theta^2+n\theta-a=0 \ldots .(4)$,

Then $\Sigma\alpha=m$, $\Sigma\alpha\beta=n$, and $\alpha\beta\gamma=a$. Since α is a root of (4), we have

$$\alpha^3=m\alpha^2-n\alpha+a, \text{ or}$$
$$\alpha^4=m\alpha^3-n\alpha^2+a\alpha=(m^2-n)\alpha^2-(mn-a)\alpha+am \ldots .(5).$$

Hence the product of the first factor in (2) by the first factor in (3), is $U+\alpha V+\alpha^2W$, where

$$U=uu'+a(vw'+v'w)+amww',$$
$$V=uv'+u'v-n(vw'+v'w)-(mn-a)ww',$$
$$W=uw'+u'w+vv'+m(vw'+v'w)+(m^2-n)ww',$$

and where the values of α^3 and α^4 were substituted from (5).

In like manner, we find that the product of the six factors in (2) and (3), to be

$$(U+\alpha V+\alpha^2W)(U+\beta V+\beta^2W)(U+\gamma V+\gamma^2W) \ldots .(6).$$

Now put $u'=u$, $v'=v$, $w'=w$; then (6) becomes $(u+\alpha v+\alpha^2 w)^2(u+\beta v+\beta^2 w)^2 \times(u+\gamma v+\gamma^2 w)^2$, where

$$U=u^2+2avw+amw^2 \ldots.(A),$$
$$V=2uv-2nvw-(mn-a)w^2 \ldots.(B),$$
$$W=2uw+v^2+2mvw+(m^2-n)w^2 \ldots.(C).$$

Whence we have the general solution of

$$U^3+m\,U^2V+(m^2-2n)U^2W+nUV^2+(mn-3a)UVW+(n^2-2am)UW^2+aV^3 +amV^2W+anVW^2+a^2W^3=z^2 \ldots.(7).$$

To make (7) applicable to (1), we set $U=x$, $V=y$, $W=0$. Then from (A), (B), and (C), we find

$$u=-\frac{v^2+2mvw+(m^2-n)w^2}{2w},$$
$$x=u^2+2avw+amw^2,$$
$$y=2uv-2nvw-(mn-a)w^2,$$

in which v and w may be chosen at pleasure. When $m=n=0$,

$$u=-\frac{v^2}{2w},\quad x=u^2+2avw,\quad y=2uv+aw^2,$$

Hence, since u and v may be chosen at pleasure, there is an infinite number of solutions corresponding to each integral value of a.

Excellent solutions were also received from *G. B. M. ZERR*, and the *PROPOSER*.

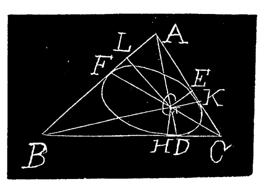
AVERAGE AND PROBABILITY.

126. Proposed by J. SCHEFFER A. M.. Hagerstown. Md.

Find the average ellipse inscribed in a triangle, so that the sides of the triangle are tangent to the ellipse.

Solution by G. B. M. ZERR. A. M.. Ph. D.. Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let ABC be the triangle; DEF the inscribed ellipse; D, E, F the points where the sides touch the ellipse; G, the point of concurrence of AD, BE, CF; $GH=l$, $GK=m$, $GL=n$; l, m, n, the perpendiculars from G on the sides; BA, BC, the axes of coördinates; a, b, c the sides of the triangle; $\triangle$, its area; P, the area of the ellipse; and Q, the average area required. Then



$$[(blm+cln)x+(aln+bmn)y-acln]^2-4b^2lm^2nxy=0,$$

is the equation to the ellipse.

$$\therefore P=\frac{\pi a^2bc^2lmn\sin B}{2\sqrt{[(ablm+acln+bcmn)^3]}}=\frac{\pi abc\triangle lmn}{\sqrt{[(ablm+acln+bcmn)^3]}}.$$

But $al+bm+cn=2\triangle$.

$$\therefore P=\frac{\pi\triangle acln[2\triangle-al-cn]}{\sqrt{[(2\triangle al+2\triangle cn-acln-a^2l^2-c^2n^2)^3]}}.$$

From the equation to the ellipse, we get

$$BD=u=\frac{acn}{bm+cn}=\frac{acn}{2\triangle-al},\quad BF=v=\frac{acl}{al+bm}=\frac{acl}{2\triangle-cn}.$$

$$\therefore al=\frac{2\triangle v(a-u)}{ac-uv},\quad cn=\frac{2\triangle u(c-v)}{ac-uv}.$$

$$\therefore P=\frac{\pi\triangle uv\sqrt{[(a-u)(c-v)]}}{\sqrt{[(av+cu-uv)^3]}}.\quad \text{Let } a-u=t,\ c-v=z.$$

$$\therefore P=\frac{\pi\triangle[a-t][c-z]\sqrt{[tz]}}{\sqrt{[(ac-tz)^3]}}.$$

$$\therefore Q=\int_0^a\int_0^c Pdtdz\Big/\int_0^a\int_0^c dtdz=\frac{\pi\triangle}{ac}\int_0^a\int_0^c\frac{(a-t)(c-z)\sqrt{(tz)}dtdz}{\sqrt{[(ac-tz)^3]}}.$$

Let $tz=ac\sin^2\theta$, $\theta'=\sin^{-1}\sqrt{(t/a)}$.

$$\therefore Q=\frac{2\pi\triangle}{a}\int_0^a\int_0^{\theta'}\frac{(a-t)(t-a\sin^2\theta)\sin^2\theta dtd\theta}{t^2\cos^2\theta}$$

$$=\frac{\pi\triangle}{a}\int_0^a\left(\frac{a-t}{t^2}\right)\{3a\sin^{-1}\sqrt{(t/a)}-2t\sin^{-1}\sqrt{(t/a)}-3\sqrt{[t(a-t)]}\}dt.$$

Let $t=a\sin^2\varphi$.

$$\therefore Q=2\pi\triangle\int_0^{\frac{1}{2}\pi}(3\varphi-2\varphi\sin^2\varphi-3\sin\varphi\cos\varphi)\cot^3\varphi d\varphi=\tfrac{1}{2}\pi^2\triangle(7-10\log^2)$$

127. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

What is the probable error of the volume of a rectangular parallelopiped whose edges measured by the repeated application of a unit of measure are found to be a, b, c, supposing that the probable error of a line so measured whose length is found to be l is $r\sqrt{l}$?

Solution by the PROPOSER.

The probable error for a, $=r\sqrt{a}$; for b, $r\sqrt{b}$; for c, $r\sqrt{c}$.

$\therefore$ Error of volume in length$=bcr\sqrt{a}$.

Error of volume in width$=acr\sqrt{b}$.

Error of volume in thickness$=abr\sqrt{c}$.

The probable error of volume=square root of the sum of the squares of these three errors.

$\therefore$ Probable error$=\sqrt{[(ab^2c^2+a^2bc^2+a^2b^2c)r^2]}=r\sqrt{[abc(ab+ac+bc)]}$.

MISCELLANEOUS.

124. Proposed by J. W. YOUNG, Graduate Student, Cornell University, Ithaca, N. Y.

Prove that the general value of θ, which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\dots\text{to } n \text{ factors}=1 \text{ is } \frac{4m\pi}{n(n+1)};$$

where m is any integer $(i=\sqrt{-1})$.

Solution by G. W. GREENWOOD, A. M., McKendree College, Lebanon. Ill.; LON C. WALKER, A. M., Leland Stanford Jr. University, Cal., and J. SCHEFFER, A. M., Hagerstown, Md.

$$1=(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\dots(\cos n\theta+i\sin n\theta)=(\cos\theta+i\sin\theta)^{1+2+\dots+n}$$

$$=(\cos\theta+i\sin\theta)^{\frac{1}{2}[n(n+1)]}=\cos\frac{n(n+1)\theta}{2}+i\sin\frac{n(n+1)\theta}{2}.$$

$$\therefore \frac{n(n+1)\theta}{2}=2m\pi, \text{ where } m \text{ is any integer; } i.\ e.\ \theta=\frac{4m\pi}{n(n+1)}.$$

Also solved by *G. B. M. ZERR.*

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Assume $m=nt+\varepsilon-\omega$, thus giving $v=m+e\sin v$ as the relation connecting the mean and eccentric anomalies, then express $x=a\cos v$, $y=b\sin v$, and $r=a(1-e\cos v)$ by a Fourier series in terms of m.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $y_1=z+x\varphi((y)$, we get by Lagrange's Theorem,

$$f(y_1)=f(z)+x\varphi(z)f'(z)+\frac{x^2}{1.2}\frac{d}{dz}\{[\varphi(z)]^2f'(z)\}+\frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\{[\varphi(z)]^3f'(z)\}+\text{etc., etc.}$$

From $v=m+e\sin v$, $y_1=v$, $z=m$, $x=e$, $\varphi(y)=\sin v$.

Now $f(v)=v$ and $f'(v)=1$.

$$\therefore\ y_1=z+x\sin z+\frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z)+\frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2(\sin^3 z)+\text{ etc.}$$

$$\therefore\ y_1=z+x\sin z+\frac{x^2}{1.2}\frac{d}{dz}\left(\frac{1-\cos 2z}{2}\right)+\frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\left(\frac{3\sin z-\sin 3z}{4}\right)$$

$$+\frac{x^4}{1.2.3.4}\left(\frac{d}{dz}\right)^3\left(\frac{3-4\cos 2z+\cos 4z}{8}\right)+\text{ etc.}$$

$$=z+x\sin z+\tfrac{1}{2}x^2\sin 2z+\tfrac{1}{8}x^3(3\sin 3z-\sin z).$$

$$\therefore\ v=m+e\sin m+\tfrac{1}{2}e^2\sin 2m+\tfrac{1}{8}e^3(3\sin 3m-\sin m)$$

$$+\tfrac{1}{6}e^4(2\sin 4m-\sin 2m)+\text{ etc. }=m+e\sin v.$$

$$\therefore\ \sin v=\sin m+\tfrac{1}{2}e\sin 2m+\tfrac{1}{8}e^2(3\sin 3m-\sin m)+\tfrac{1}{6}e^3(2\sin 4m-\sin 2m)+\text{etc.}$$

To develop $(1-e\cos v)$ in terms of m: Let $f(y_1)=1-e\cos y_1$, $f'(y_1)=e\sin y_1$.

$$\therefore\ 1-e\cos y_1=(1-e\cos z)+x\sin z(e\sin z)+\frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z.e\sin z)+\text{ etc.}$$

Performing the operations as before we get after substituting v for y_1, m for z, e for x,

$$1-e\cos v=1-e\cos m+\tfrac{1}{2}e^2(1-\cos 2m)+\tfrac{1}{8}e^3(3\cos m-3\cos 3m)$$

$$+\tfrac{1}{4}e^4(\cos 4m-\cos 2m)+\text{ etc.}$$

$$\therefore\ \cos v=\cos m+\tfrac{1}{2}e(\cos 2m-1)+(3e^2/8)(\cos 3m-\cos m)$$

$$+\tfrac{1}{4}e^3(\cos 2m-\cos 4m)+\text{ etc.}$$

$$\therefore\ \sin v=A\sin m+B\sin 2m+C\sin 3m+D\sin 4m+\dots.$$
$$\cos v=-\tfrac{1}{2}e+A_1\cos m+B_1\cos 2m+C_1\cos 3m+D_1\cos 4m+\dots.$$

where A, B, C, D, ...A_1, B_1, C_1, D_1, ... are each a series in powers of e.

$$\therefore\ x=a\cos v=-\tfrac{1}{2}ae+aA_1\cos m+aB_1\cos 2m+aC_1\cos 3m+aD_1\cos 4m+\dots.$$
$$y=b\sin v=bA\sin m+bB\sin 2m+bC\sin 3m+bD\sin 4m+\dots.$$
$$r=a(1-e\cos v)=a[1+\tfrac{1}{2}e^2-eA_1\cos m-eB_1\cos 2m-eC_1\cos 3m$$
$$-eD_1\cos 4m+\dots.].$$

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The declination of a certain fixed star if 12° 40′. Its altitude was observed one day to be 16° 40′. Three hours and twenty-four minutes later it was found to be 40° 20′. Find the latitude of the place of observation.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\delta=12^\circ\ 40'$, φ=latitude of observation, $a=16^\circ\ 40'$, $a'=40^\circ\ 20'$, $\mu=51^\circ$ $=3$ hours, 24 minutes, h=hour angle.

Then $\cos h=\dfrac{\sin a-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}=x$, $\cos(h-\mu)=\dfrac{\sin a'-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}=y$.

Eliminating h, $\sin^2\mu=x^2-2xy\cos\mu+y^2$. Substituting values of x and y,

$$\cos^2\varphi\cos^2\delta\sin^2\mu=2\sin^2\varphi\sin^2 d(1-\cos\mu)+\sin^2 a+\sin^2 a'-2\sin a\sin a'\cos\mu$$
$$-2\sin\varphi\sin\delta(1-\cos\mu)(\sin a+\sin a').$$

$$\therefore\ \sin^2\varphi[\sin^2\mu+\sin^2\delta(1-\cos\mu)^2]-2\sin\varphi\sin\delta(1-\cos\mu)(\sin a+\sin a')$$
$$=\cos^2\delta\sin^2\mu-\sin^2 a-\sin^2 a'+2\sin a\sin a'\cos\mu.$$

Let $\sin^2\mu+\sin^2\delta(1-\cos\mu)^2=A=.610569$.
$\sin\delta(1-\cos\mu)(\sin a+\sin a')=B=.075921$.
$\cos^2\delta\sin^2\mu-\sin^2 a-\sin^2 a'+2\sin a\sin a'\cos\mu=C=.307394$.

$\therefore\ A\sin^2\varphi-2B\sin\varphi=C$.

$\therefore\ \sin\varphi=\dfrac{B\pm\sqrt{(AC+B^2)}}{A}=.844702$, or $-.596013$. $\varphi=57^\circ\ 38'\ 37''$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D.. Professor of Mathématics and Astronomy in Defiance College, Defiance, Ohio.

If I sell one of my farms for $\$A$,=\$4500, and the other for $\$B$,=\$1800, I will gain p%, =5%, on cost of both; but if I sell the dearer farm for $\$C$,=\$4000, and the other at cost, I will lose p%,=5%. Find the cost of each farm.

AVERAGE AND PROBABILITY.

138. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of (1) triangle, (2) quadrilateral, (3) pentagon, (4) hexagon, formed by taking (1) three, (2) four, (3) five, (4) six random points on the circumference of a given circle radius a.

139. Proposed by L. C. WALKER, A. M., Graduate Student. Leland Stanford Jr. University, Cal.

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

MISCELLANEOUS.

131. Proposed by SAUL EPSTEEN, Ph. D., Professor of Mathematics, University of North Carolina.

Find a power series for π^{nx} (n=any integer).

132. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Six officers of different grades (1, 2, 3, 4, 5, 6) from each of six branches of the army (a, b, c, d, e, f) are to be arranged in a square so that each rank and each file shall have an officer of each grade and each branch. Can it be done? If not, prove it. The arrangement of five officers of each kind is easy.

133. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a group G of order mn has a subgroup H of order n, and if n has no prime factor which is less than m, show that H must be a self-conjugate sub-group. (Frobenius.)

134. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Give a complete solution of the Jacobian equation $\kappa^2\text{sn}^4u+2\kappa^2\text{sn}^2u+1=0$.

BOOKS AND PERIODICALS.

Accounting and Business Practice. By John H. Moore, Commercial Department, Boston High School, and George W. Miner, Commercial Department, Westfield (Mass.) High School. 8vo. Cloth, 400 pages. List price, $1.50; mailing price, $1.55. Boston: Ginn & Co.

"Accounting and Business Practice is a thorough, practical, and comprehensive text for the use of teachers and students of book-keeping. It is intended for use in high schools, private schools, and all institutions where accounting is taught, and is well adapted for teaching by correspondence. Attractive blank books and business forms accompany the text. The work is arranged in the following general divisions: Introductory, presenting a series of definite lessons for beginners embracing lesson outlines, exercises for class drills, two brief sets in elementary accounting, and two sets for business practice. Intermediate, presenting the subject of drafts, three sets of more advanced business practice, introducing the use of special columns, and auxiliary ledgers. Advanced, containing three sets, single entry, corporation accounting (a set on manufacturing), and banking.

A few special features: 1. The work is complete in itself and is not accompanied by a system of vouchers. 2. The work is elastic and may be used in the study of theory only, or of theory and business practice. 3. Financial statements are given in connection with all the different sets. 4. Class exercises are given in connection with every important subject introduced. 5. The text is accompanied by a Teachers' Manual giving a large amount of material for class drills in practical accounting, arranged in a series of lessons carefully graded."

The Universal Solution for Numerical and Literal Equations by which the Roots of Equations of All Degrees can be expressed in terms of their Coefficients. By M. A. McGinnis. 8vo. Cloth, 194 pages. Price, $2.00. Kansas City: The Mathematical Book Co.

We cannot praise this book very highly for the merit it possesses, since the really meritorious part of the book deals with matter quite irrelevant to what the work professes

to discuss, and to solve, viz., the solution of the general equation of the fifth and other degrees. The book contains some ingeneous methods of solving certain numerical equations, but because of these methods it should have received a more modest title.

Mr. McGinnis's solution of the Sixth Degree is quite erroneous. Mr. W. M. H. Woodward, pp. 153-150, professes to have demolished the proof of the impossibility of solving the general quantic by radicals given in Serret's *Algebra Supérieure.* But judging from his conclusion, it appears that Mr. Woodward does not understand the argument put forth in *Algebra Supérieure.* B. F. F.

An Elementary Text-book on the Differential and Integral Calculus. By William H. Echols, Professor of Mathematics in the University of Virginia. 8vo. Cloth, x+480 pages. Price, $2.00. New York: Henry Holt & Co.

In this work are very ably treated many interesting subjects not to be found in any other American text-book on the Calculus.

In order to form a connecting link between Algebra and the Calculus, an Introduction presents in an admirable way the fundamental and essential features of Arithmetic and Algebra. In the Introduction are defined and explained such ideas as *absolute number, the absolute-number continuum, the real-number system, the limit of a variable,* etc., ideas upon which rest the whole structure of the Calculus. Throughout the work, in establishing the principles much attention is given to the applications of those principles. An unusually large number of interesting and well selected problems are appended to each section.

The work is divided into two books. Book I treats of functions of one variable, and is divided into four parts. Part I embraces the Principles of the Differential Calculus; Part II applies these principles to Geometry; Part III establishes the Principles of the Integral Calculus, and Part IV embraces the application of these principles. Book II treats of functions of more than one variable. It is divided into three parts. The first part which is Part V of the entire work, embraces Principles and Theory of Differentiation; Part VI applies the principles to surfaces, and Part VII treats of Integration of more than one Variable and Multiple Integration.

Part VI extends the principles of the Calculus to surfaces. Here we have such problems as: To find the principal radii to a surface; To determine the umbilics on a surface, etc. Also here is discussed pretty fully such subjects as spherical curvature, envelopes of surfaces, etc.

The author, while acknowledging that the introduction of a new symbolism is always objectionable, yet feels called upon to introduce the "English pound" mark for the symbol of passing to the limit. This is certainly desirable. But personally we prefer Professor Oliver's symbol, $\doteq$, for "converging to" or approaches, to Professor Echol's symbol, (=), which he introduces to mean the same thing.

The work is a most valuable addition to the many meritorious books on the same subject which have appeared in recent years. B. F. F.

The School Visitor. Published by John S. Royer & Sons, No. 247 North 17th Street, Columbus, Ohio. Price per year, $1.00, payable in advance.

The Mathematical Department is full of good problems for the teacher of Arithmetic, Algebra and Geometry. Mr. Royer is the author of a Higher Mental Arithmetic, a Geography, and several other books of great interest and value to teachers. B. F. F.

ERRATA.

Vol. IX, page 207, problem 106, Diophantine Analysis, for "rational triangle" read, *rational right triangle.*

Vol. IX, page 264, last line of solution of problem 100, for "determinate" read, *indeterminate*; page 265, equation for "$Ay'q+$" read $Ay'+q$.

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THE GROUP GENERATED BY CENTRAL SYMMETRIES, WITH APPLICATION TO POLYGONS.

By DR. EDWARD KASNER.

The object of this article is to generalize the following well known theorem of elementary geometry: If the mid-points of the consecutive sides of any quadrilateral are joined the resulting figure is a parallelogram. In the case of a triangle the corresponding construction gives a triangle not having any peculiar property. The question therefore arises as to whether or not in case of polygons of more than four sides there is any theorem analogous to that concerning the quadrilateral. It will be shown that, in this respect, there is an essential distinction between polygons of an even number of sides, and those of an odd number of sides. This depends fundamentally upon the character of the group which is discussed in §1.

§1. THE GROUP.

1. Any translation T of the plane may be written

$$(T)\qquad \begin{aligned} x' &= x+h \\ y' &= y+k, \end{aligned}$$

where h and k are the components, in the direction of the coördinate axes, of the vector corresponding to the translation. It is obvious that the totality of translations form a group; for the combination of any two, say T_1 and T_2 whose vector components are h_1, k_1 and h_2, k_2 respectively, gives

$$x'=x+h_1+h_2$$
$$y'=y+k_1+k_2,$$

which is itself a translation.

2. Consider now the transformations termed *central or point symmetries.* Such a symmetry is defined by

$$(S) \qquad \begin{aligned} x'&=-x+2a \\ y'&=-y+2b, \end{aligned}$$

where a, b are the coördinates of the center of the symmetry, *i. e.* the fixed point P with respect to which corresponding points x, y and x', y' are symmetric. The symmetries themselves do not form a group, but we now show that

The translation T and the central symmetries S form a group.

In the first place, *the product of two symmetries is a translation.* For, if the center of S_1 is P_1, with coördinates (a_1, b_1), and if the center of S_2 is P_2, with coördinates (a_2, b_2), the combination $S_1 S_2$ gives

$$x'=x+2(a_2-a_1)$$
$$y'=y+2(b_2-b_1).$$

The vector of this translation is double the vector $P_1 P_2$. Similarly, the produtc $S_2 S_1$ is the translation whose vector is twice $P_2 P_1$. In the second place, *the product of a symmetry and a translation is a symmetry.* For, the transformation ST is

$$x'=-x+2a+h$$
$$y'=-y+2b+k;$$

this is the symmetry whose center is obtained from the center of S by applying the vector of T. Similarly, the product in the reverse order, *i. e.*, TS is the symmetry whose center is obtained from the center of S by applying the vector opposite to that of T.

It follows then that any combination of transformations T and S is itself either a T or an S, so that the group property is proved. In Lie's terminology the group considered is a mixed two-parameter group consisting of two continuous systems of transformations. The translations constitute a self-conjugate sub-group.

3. Since the product of two symmetries is a translation, and since the translations constitute a group, it follows that *the product of an even number of symmetries is a translation.* The product of the $2k$ symmetries S_1, S_2, ..., S_{2k} is in fact

$$x'=x+2(a_{2k}-a_{2k-1}+\dots+a_2-a_1)$$
$$y'=y+2(b_{2k}-b_{2k-1}+\dots+b_2-b_1),$$

where a_i, b_i denote the caördinates of P_i the center of S_i. The formulae may be

interpreted geometrically by observing that the differences a_2-a_1, b_2-b_1, for example, are the components of the vector P_1P_2; therefore *the vector of the resulting translation is twice the vector sum*

$$P_1P_2+P_3P_4+\dots+P_{2k-1}P_{2k}.$$

4. An odd number of symmetries may be combined by combining the first with the product of all the remaining, which by 3 is a translation. Therefore, *the product of an odd number of symmetries is a symmetry.* If the symmetries are S_1, S_2, ..., S_{2k+1} their product is

$$x'=-x+2(a_{2k+1}-a_{2k}+\dots+a_3-a_2+a_1)$$
$$y'=-y+2(b_{2k+1}-b_{2k}+\dots+b_3-b_2+b_1).$$

The center of the resulting symmetry is obtained from the center P of the first symmetry by applying the vector sum

$$P_2P_3+P_4P_5+\dots+P_{2k}P_{2k+1}.$$

5. The application to be made depends essentially upon the *fixed points* of the transformations, *i. e.* the points which are transformed into themselves. Excluding points at infinity from consideration, we observe in the first place that in case of a symmetry there is one and but one fixed point, namely, the center of the symmetry. On the other hand, in case of a translation, there are no fixed points, except when the translation reduces to the identical transformation, in which case all the points of the plane are fixed.

§2. Mid-Point Polygons.

6. Consider any polygon whose vertices in order may be denoted by Q_1, Q_2, ..., Q_n. If the middle points of the sides are connected in order we derive a new polygon of the same number of sides which for brevity may be termed the *inscribed polygon*; the original polygon in its relation to the derived is then termed the *circumscribed polygon.* For every polygon there is then a definite inscribed polygon. The question now to be considered concerns the converse problem: *Given an arbitrary polygon P_1, ..., P_n, is it possible to construct a circumscribed polygon, i. e.*, is it possible to find n points Q_1, Q_2, ..., Q_n such that P_1 shall be mid-way between Q_1 and Q_2, P_2 mid-way between Q_2 and Q_3, and so on until finally P_n shall be mid-way between P_n and P_1?

To answer this question, take tentatively any point Q in the plane; construct with respect to P_1 the symmetric point; then with respect to P_2 construct the point symmetric to the one just obtained; and so in order until finally by symmetry with respect to P_n a point Q' is obtained. The original polygon P_1, ..., P_n thus defines a definite transformation by which to any point Q corresponds an unique point Q'. This transformation is simply the product of n symmetries

and therefore, by the previous section, is either a translation or a symmetry according as n is even or odd.

7. Consider first the case of a polygon with an odd number of sides $n=2k+1$. The transformation from Q to Q' is then a symmetry. There is, therefore, by §5 a single point which remains invariant under the transformation. Denoting this point by Q_1, we obtain from it, by successive symmetry with respect to P_1, P_2, ..., P_{2k}, the points Q_2, Q_3, ..., Q_{2k+1}; Q_{2k+1} and Q_1 are then necessarily symmetric with respect to the last vertex P_{2k+1}, so that Q_1, ..., Q_{2k+1} are in fact the vertices of the unique circumscribed polygon.

Any polygon with an odd number of sides can be obtained as an inscribed polygon; there exists one and only one circumscribed polygon.

The circumscribed polygon may be constructed by applying the result stated at the end of 4. The first vertex Q_1 is obtained from the first vertex P_1 of the original polygon by applying the vector sum $P_2P_3+P_4P_5+....+P_{2k}P_{2k+1}$; then the remaining vertices Q_2, ..., Q_{2k+1}, are obtained by successive symmetry as described above.

8. If the polygon has an even number of sides $n=2k$, then from 3 the transformation from Q to Q' is a translation which in general does not reduce to the identical transformation. In this case, from 5, there exists no fixed point, and therefore no circumscribed polygon.

In the case of an arbitrary polygon of an even number of sides no circumscribed polygon exists, i. e., not all such polygons can be obtained as inscribed polygons.

9. The construction will, however, be possible in the exceptional case where the resulting translation reduces to identity. If then we term a $2k$-gon *special* when it is possible to circumscribe a polygon about it, the result may be stated:

Any special $2k$-gon is characterized by the fact that the product of the symmetries having for centers the vertices of the polygon is identity.

The class of polygons considered may be defined otherwise as follows: From the formulae in 3, we have as the conditions for reducing to identity,

$$a_{2k}-a_{2k-1}+....+a_2-a_1=0$$
$$b_{2k}-b_{2k-1}+....+b_2-b_1=0;$$

which together express the vanishing of the vector sum

$$P_1P_2+P_3P_4+....P_{2k-1}P_{2k}.$$

The vanishing of this sum necessitates the vanishing of

$$P_2P_3+P_4P_5+....+P_{2k}P_1,$$

since for any polygon the vector sum of all the sides is zero. Therefore

In any special $2k$-gon the vector sum of the alternate sides vanish; this condition is also sufficient.

The equation of condition above may also be written

$$\frac{a_1+a_3+\ldots+a_{2k-1}}{k}=\frac{a_2+a_4+\ldots+a_{2k}}{k}$$

$$\frac{b_1+b_3+\ldots+b_{2k-1}}{k}=\frac{b_2+b_4+\ldots+b_{2k}}{k},$$

which may be interpreted as follows:

In any special 2k-gon with vertices $P_1, P_2, \ldots, P_{2k}$, *the mean point (or center of gravity) of the alternate vertices* $P_1, P_3, \ldots, P_{2k-1}$ *coincides with the mean point of the remaining vertices* $P_2, P_4, \ldots, P_{2k}$.

Both points obviously coincide with the mean point of all the vertices of the $2k$-gon.

10. For a special $2k$-gon the transformation from Q to Q' described in 6 reduces to identity, so that every point of the plane is an invariant point. Therefore in constructing the circumscribed polygon any point may be assumed for the first vertex Q_1, the other vertices then being determined by successive symmetry with respect to $P_1, P_2, \ldots, P_{2k-1}$.

About a special 2k-gon a double-infinity of circumscribed 2k-gons may be constructed; otherwise stated, if it is possible to circumscribe one polygon about a given 2k-gon, it is possible to circumscribe a double infinity.

We shall now prove that among this double infinity of $2k$-gons there is one which is itself special, so that *abont any special 2k-gon it is possible to circumscribe one and only one special 2k-gon.* Let the first vertex Q_1 of any circumscribed polygon be x, y; then the next vertex Q_2, obtained by symmetry with respect to P_1, is $-x+2a_1$, $-y+2b_1$; similarly Q_2 is $x-2a_1+2a_2$, $y-2b_1+2b_2$; finally, Q_{2k} is $-x+2a_1-2a_2+\ldots+2a_{2k-1}$, $-y+2b_1-2b_2+\ldots+2b_{2k-1}$. If now the circumscribed polygon is to be special we must have

$$Q_1Q_2+Q_3Q_4+\ldots+Q_{2k-1}Q_{2k}=0,$$

which is equivalent to

$$kx-(2k-1)a_1+(2k-2)a_2-\ldots-a_{2k-1}=0$$
$$ky-(2k-1)b_1+(2k-2)b_2-\ldots-b_{2k-1}=0$$

These equations determine x and y, that is, the first vertex Q_1, uniquely, which proves the theorem announced.

The special quadrilaterals are simply parellelograms. About any parallelogram a double infinity of quadrilaterals may be circumscribed, of which one is itself a parallelogram; about this in turn a parallelogram may be circumscribed and so on indefinitely. So for any special $2k$-gon, one can not only inscribe special $2k$-gons indefinitely, but also circumscribe them. Again, just as the quadrilateral inscribed in a general parallelogram is itself an arbitrary parallelogram, so the $2k$-gon inscribed in a special $2k$-gon is not further specialized, but is an arbitrary special $2k$-gon.

11. The second characteristic given in 9 gives the following *construction for special* $2k$*-gons.* Take an arbitrary k-gon D', D'', ..., $D^{(k)}$; on each side construct a parallelogram, thus on $D'\ D''$ construct $P_1 D_1 D_2 P_2$, on $D''\ D'''$ construct $P_3 D'' D''' P_4$, finally construct $P_{2k-1} D^{(k-1)} D^{(k)} P_{2k}$; then $P_1, P_2, ..., P_{2k}$ will constitute a special $2k$-gon. To prove this we need merely observe that since the alternate sides $P_1\ P_2, P_3\ P_4$, ..., are equal and parallel to $D'\ D''$, $D''\ D'''$, respectively, the vector sum of the former sides is equal to the vector sum of all the sides of the auxiliary k-gon and therefore vanishes.

It is seen from this that a *special* $2k$*-gon is completely determined by* $2k-1$ *of its vertices.* For if P_1, P_2, ..., P_{2k-1} are given we can construct the auxiliary k-gon by starting at an arbitrary point D', drawing the vector $D'D''$ equal to P_1P_2, then $D''D'''$ equal to P_3P_4, ..., finally, $D^{(k-1)}D^{(k)}$ equal to $P_{2k-3}P_{2k-2}$; P_{2k} is then found by drawing from P_{2k-1} a vector equal to $D^{(k)}D'$. This is the generalization of the fact that a parallelogram is determined by three of its vertices (given of course in order).

After the case $k=2$ of the parallelogram, the first case deserving particular attention is the case $k=3$, *i. e.*, the *special hexagon.* Such a hexagon may be obtained, in accordance with the result above, by constructing parallelograms on the sides of an arbitrary triangle. Another construction is as follows: Take any two parallelograms $ABCO$, $ODEF$ having a vertex in common; the remaining vertices $ABCDEF$ constitute a special hexagon. The same hexagon may be obtained in this way by means of three distinct pairs of parallelograms. This may be generalized so as to apply to $2k$-gons.

The third characteristic stated in 9, in the case of a special hexagon $ABCDEF$, shows that the median point of the triangle ACE coincides with that of the triangle BDF; therefore the six lines obtained by joining each vertex to the mid-point of the opposite diagonal of the hexagon are concurrent, the point of concurrence being the mean point of the hexagon.

§3. Extension to Space.

12. The preceding results admit of immediate extension to space of three dimensions, and to higher spaces. In fact, the translations and central symmetries still constitute a group, and results for polygons in space follow in a manner entirely analogous to that employed above. One difference as to the character of point symmetries may be noticed: In the plane a point symmetry is identical with a rotation of the plane in itself through 180°; in space however point symmetry is not equivalent to a rotation since in fact corresponding figures are not congruent but differ in the order of arrangement of their parts. If we consider a central symmetry in the plane as a rotation, the analogue in space would be a line symmetry, which is in fact rotation about an axis through 180°. However, in the application to space polygons only the symmetries of the former type, *i. e.* point symmetries with respect to the vertices, are considered.

The results hold also for one dimension, that is, for sets of points in a line. Thus, for any set P_1, P_2, ..., P_n there is a derived "inscribed" set consist-

ing of the mid-points of the segments, P_1P_2, P_2P_3, ..., P_nP_1. This set is entirely arbitrary if n is odd, but not if n is even; the characteristics stated in 9 apply almost literally to these special sets of points of the latter type.

COLUMBIA UNIVERSITY, NEW YORK, *January 15, 1903.*

ANALOG OF SYLVESTER'S DIALYTIC METHOD OF ELIMINATION.

By DR. SAUL EPSTEEN.

If the two equations

$$a_0x^3+a_1x^2+a_2x+a_3=0,$$
$$b_0x^2+b_1x+b_2=0,$$

have a root x_1 in common, we can write

$$\begin{aligned}
a_0x_1^4+a_1x_1^3+a_2x_1^2+a_3x_1 &\equiv 0\\
a_0x_1^3+a_1x_1^2+a_2x_1+a_3 &\equiv 0\\
b_0x_1^4+b_1x_1^3+b_2x_1^2 &\equiv 0\\
b_0x_1^3+b_1x_1^2+b_2x_1 &\equiv 0\\
b_0x_1^2+b_1x_1+b_2 &\equiv 0.
\end{aligned}$$

Eliminating x_1^4, x_1^3, x_1^2, x_1, 1, we obtain

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0\\ 0 & a_0 & a_1 & a_2 & a_3\\ b_0 & b_1 & b_2 & 0 & 0\\ 0 & b_0 & b_1 & b_3 & b_1\\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix}=0.$$

This is the well known dialytic method of elimination of Sylvester.

We can deal with linear differential equations in exactly the same manner. Suppose there are given the equations

$$(1)\qquad \alpha_0(x)\frac{d^3y}{dx^3}+\alpha_1(x)\frac{d^2y}{dx^2}+\alpha_2(x)\frac{dy}{dx}+\alpha_3(x).y=0,$$

$$(2)\qquad \beta_0(x)\frac{d^2y}{dx^2}+\beta_1(x)\frac{dy}{dx}+\beta_2(x).y=0.$$

If these equations have an integral y_1 in common we can, by differentiation, write

$$\alpha_0 \frac{d^4 y_1}{dx_4} + (\alpha_0' + \alpha_1) \frac{d^3 y_1}{dx^3} + (\alpha_1' + \alpha_2) \frac{d^2 y_1}{dx^2} + (\alpha_2' + \alpha_3) \frac{dy_1}{dx} + \alpha_3' y_1 \equiv 0.$$

$$\alpha_0 \frac{d^3 y_1}{dx^3} + \qquad \alpha_1 \frac{d^2 y_4}{dx^2} + \qquad \alpha_2 \frac{dy_1}{dx} + \alpha_3 y_1 \equiv 0.$$

$$\beta_0 \frac{d^4 y_1}{dx^4} + (2\beta_0' + \beta_1) \frac{d_3 y_1}{dx^3} + (\beta_0'' + 2\beta_1' + \beta_2) \frac{d^2 y_1}{dx^2} + (\beta_1'' + 2\beta_2') \frac{dy_1}{dx} + \beta_2'' y_1 \equiv 0.$$

$$\beta_0 \frac{d^2 y_1}{dx^2} + \qquad (\beta_0' + \beta_1) \frac{d^2 y_1}{dx^2} + \qquad (\beta_1' + \beta_2) \frac{dy_1}{dx} + \beta_2' y_1 \equiv 0.$$

$$\beta_0 \frac{d^2 y_1}{dx^2} + \qquad \beta_1 \frac{dy_1}{dx} + \beta_2 y_1 \equiv 0.$$

Whence, eliminating $\frac{d^4 y_1}{dx^4}$, $\frac{d^3 y_1}{dx^3}$, $\frac{d^2 y_1}{dx^2}$, $\frac{dy_1}{dx}$, y_1, we find that

$$\begin{vmatrix} \alpha_0 & (\alpha_0' + \alpha_1) & (\alpha_1' + \alpha_2) & (\alpha_2' + \alpha_3) & \alpha_3' \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & (2\beta_0' + \beta_1) & (\beta_0'' + 2\beta_1' + \beta_2) & (\beta_1'' + 2\beta_2') & \beta_2'' \\ 0 & \beta_0 & (\beta_0' + \beta_1) & (\beta_1' + \beta_2) & \beta_2' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} = 0,$$

for all values of x.

The generalization to equations of order m and n is of course easy.

This theorem is not to be found in any of the standard treatises or text-books. Indeed it is rather surprising that Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, which is such a complete treatise on linear differential equations, does not contain it. The theorem was proved by Professor von Escherisch in the *Denkschriften der Wiener Akademie*, Vol. 46. However, these *Denkschriften* have but a very small circulation in America and the volumes preceeding the 64th seem to be particularly rare. Even such complete collections as the University of Chicago Library and the John Crerar Library (Chicago) do not have them. Possibly this accounts for the oblivion in which Professor von Escherisch's theorem has hitherto rested. Personally I have never been able to get a copy of Vol. 46 of the *Denkschriften der Wiener Akademie* to consult the paper at first hand and know of it only through the reference given by Heffter in Crelle's *Journal*, Vol. 116.

THE UNIVERSITY OF CHICAGO, *December, 1902.*

SUFFICIENT CONDITION THAT TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS SHALL HAVE COMMON INTEGRALS.

By A. B. PIERCE.

As a continuation of the preceding paper I propose to prove that the condition $\triangle=0$ for all values of x is sufficient in order that two linear homogeneous differential equations shall have common integrals. I shall follow a method analogous to that used for the corresponding problem in the Algebra as given by de Comberousse.* Although the problem has been discussed heretofore, by von Escherich and Heffter, I believe this method has not been used before.

Using the theorem that a linear homogeneous differential equation of the nth order has n and only n linearly independent integrals, I prove a sufficient condition expressed in linear homogeneous differential operators and then prove that we can obtain a form of the same character from the above mentioned condition.

Represent a differential operator of order m as follows:

$$A\equiv a_0\frac{d^m}{dx^m}+a_1\frac{d^{m-1}}{dx^{m-1}}+\ldots.+a_{m-1}\frac{d}{dx}+a_m,$$

and by B, R, S similar operators of order n, p, q, respectively. Further, RAy shall indicate that we operate upon y with A and then upon the result obtained with R; the order of the result will be $m+p$.

LEMMA: *If two linear homogeneous differential operators A and B are such that $RAy+SBy\equiv0$, where $p<n$, $q<m$, then the differential equations $Ay=0$ and $By=0$ have at least one integral in common.*

Let $RAy+BSy\equiv0$; then $m+p=n+q$.

We will carry out our proof for $m=2$, $n=3$.

Represent independent integrals of $Ay=0$ by $y_{a,1}$, $y_{a,2}$, and similarly for B, R, and S. Now considering the differential equations

$$RAy=0 \text{ and } SBy=0,$$

we know from our hypothesis that all the integrals which satisfy one of them must satisfy the other. Then $y_{b,1}$, $y_{b,2}$, $y_{b,3}$ must satisfy $RAy=0$, or $Ay_{b,1}$, $Ay_{b,2}$, $Ay_{b,3}$ satisfy the equation $Ry=0$ where $p\leqq2$.

Since we cannot have more independent integrals of the equation than its order, one of the following possibilities must be true; either

(1) One or more of the expressions Ay_b is zero; or,

(2) One is a linear combination of the other two.

*Algebre Superieure, Vol. II, Second Edition, Paris, 1887-1890.

The first case immediately gives us our theorem. In the second case we have $l_1 Ay_{b1}+l_2 Ay_{b2}+l_3 Ay_{b3}=0$, where l_1, l_2, l_3 are constants. This, however, we may write

$$Al_1 y_{b1}+Al_2 y_{b2}+Al_3 y_{b3}=A(l_1 y_{b1}+l_2 y_{b2}+l_3 y_{b3})=0.$$

That is, $l_1 y_{b1}+l_2 y_{b2}+l_3 y_{b3}$ is an integral of $Ay=0$. Since it is also an integral of $By=0$, by hypothesis, our theorem is proved. By similar reasoning we can prove the following

COROLLARY: *$Ay=0$, and $By=0$, have, at least, $m-q=n-p$ independent integrals in common.*

THEOREM. *$\triangle$ equal zero for all values of x is a sufficient condition that $Ay=0$ and $By=0$ shall have at least one integral in common.*

Here, again, we limit ourselves to the case $m=2$, $n=3$, indicating the generalization for the general case at the close of the proof. Then

$$\triangle \equiv \begin{vmatrix} \alpha_0 & 2\alpha_0'+\alpha_1 & \alpha_0''+2\alpha_1'+\alpha_2 & \alpha_1''+2\alpha_2' & \alpha_2'' \\ 0 & \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2' \\ 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 & \beta_3' \end{vmatrix}$$

We will associate with each column of $\triangle$ a definite derivative of y which we will later use as a factor in connection with the elements of that column.

Starting from the right with the first column, we will use y itself or the zero derivative; with the second column the first derivative, and so on in succession, in general with the kth column, the $(k-1)$st derivative. Let $\triangle$ vanish for all values of x.

We will now divide the proof into two parts and carry each out separately although the method used in the second part includes the first as a special case.

1. All first minors of $\triangle$ not zero.
2. All first minors of $\triangle$ zero.

1. Let the last column be one for which the minors are not all zero. Then multiplying the last column by y and adding to it all the other columns each multiplied by its corresponding derivative, and representing these sums by the letters A and B with subscripts, we obtain the following equation true for all values of x whatever y may be:

$$\begin{vmatrix} \alpha_0 & 2\alpha_0'+\alpha_1 & \alpha_0''+2\alpha_1'+\alpha_2 & \alpha_1''+2\alpha_2' & A_2 \\ 0 & \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & A_1 \\ 0 & 0 & \alpha_0 & \alpha_1 & A_0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & B_0 \\ \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 & B_1 \end{vmatrix} \equiv 0.$$

However, $A_0=Ay$, $A_1=\frac{d}{dx}Ay$, $A_2=\frac{d^2}{dx^2}Ay$, and similarly, $B_0=By$, $B_1=\frac{d}{dx}By$.

Expanding according to the elements of the last column, and calling the corresponding cofactors M and N, respectively, with subscripts in the inverse order, we have

$$M_0\frac{d^2}{dx^2}Ay+M_1\frac{d}{dx}Ay+M_2Ay+N_1By+N_0\frac{d}{dx}By\equiv 0.$$

Writing $R\equiv M_0\frac{d^2}{dx^2}+M_1\frac{d}{dx}+M_2$, and $S\equiv N_0\frac{d}{dx}+N_1$, we have $RAy+SBy\equiv 0$, where $p=2$ and $q=1$.

If the elements of the last columns have their minors zero, we proceed in the same way with some other column for which the hypothesis is satisfied.

2. In this case the matrix obtained by cutting off the first column and the first and last rows of $\triangle$, will vanish, *i. e.*, all the determinants of highest order will vanish.

Assuming that the minor determinants obtained from the elements of the first two columns do not all vanish, we will take those determinants of the matrix which have these two columns for their first two columns, and multiplying each by the derivative corresponding to the last column, we will add them all together. This gives

$$\begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & (\alpha_1'+\alpha_2)\frac{dy}{dx}+\alpha_2 y \\ 0 & \alpha_0 & \alpha_1\frac{dy}{dx}+\alpha_2 y \\ \beta_0 & \beta_1 & \beta_2\frac{dy}{dx}+\beta_3 y \end{vmatrix}\equiv 0.$$

Adding the first two columns, each multiplied by its corresponding derivative to the last column, we may rewrite

$$\begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & A_1 \\ 0 & \alpha_0 & A_0 \\ \beta_0 & \beta_1 & B_0 \end{vmatrix}\equiv 0,$$

where the symbols A_0, A_1, B_0 have the same meaning as before. Expanding as before, and writing $R=M_0\frac{d}{dx}+M_1$, $S=N_0$, we have $RAy+SBy\equiv 0$.

The next matrix cannot be zero, because then Ay would have all its coefficients zero, which gives a trivial case.

If the first and second columns do not fulfill the conditions imposed, take two which do, and proceed in a similar way. With this, the proof for the case $m=2$, $n=3$ is complete.

For the general case, the proceedure is similar; we have only to consider the possibility of other matrices vanishing. By using the following definition and properties, the general proof is quite evident.

Call first submatrix of $\triangle$ the matrix obtained by cutting off the first column and the first and last rows of $\triangle$; the second submatrix that obtained from the first by the same process; and so on successively.

If any submatrix is zero all the preceding submatrices and $\triangle$ itself are zero. If A and B are not identical, and neither identically zero, then all the submatrices do not vanish. We proceed as in the proof above, with the last one which is zero. From this we easily prove the following

COROLLARY: *If the kth submatrix be the first one, all of whose determinants of highest order do not vanish, then $Ay=0$, and $By=0$, have k independent integrals in common.*

On this foundation then, we find the number of independent integrals two linear homogeneous equations may have in common, and can also find the equation satisfied by these common integrals.

I add the following numerical cases as illustrations of the lemma:

1. $$Ay=\frac{d^3y}{dx^3}+(x-a)\frac{d^2y}{dx^2}+x(x-a)\frac{dy}{dx}-ax^2y,$$

$$By=\frac{d^2y}{dx^2}+(x-a)\frac{dy}{dx}-axy,$$

$$R=(1-x^2)\frac{d}{dx}+x(3-x^2),$$

$$-S=(1-x^2)\frac{d^2}{dx^2}+x(3-x^2)\frac{d}{dx}-(1-x^2)^2.$$

Then $Ay=0$ and $By=0$, have the common integral $y=e^{ax}$.

2. $$Ay=\frac{d^3y}{dx^3}+x(1+x)\frac{d^2y}{dx^2}+(1+x^2+x^3)\frac{dy}{dx}+x(x^3+2)y,$$

$$By=\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+(2x^2+1)\frac{dy}{dx}+x(2+x^2)y,$$

$$R=x(x-1)\frac{d}{dx}+(x^3-x^2-2x+1),$$

$$-S=x(x-1)\frac{d}{dx}+(x^4-x^3-2x+1).$$

The common integrals of $Ay=0$ and $By=0$ are integrals of

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=0.$$

GREIFSWALD, GERMANY, *January, 1903.*

NOTE ON THE POLAR OF A POINT AS TO A CONIC.

By GEORGE R. DEAN.

Let (x_1, y_1) be a point not on the curve, (x_2, y_2), (x_3, y_3) the intersections of a secant through (x_1, y_1) with the conic. The tangents at (x_2, y_2), (x_3, y_3) are

$$axx_2+h(x_2y+xy_2)+byy_2+2gx_2+2fy_2+c=0 \ldots. (1),$$
$$axx_3+h(x_3y+xy_3)+byy_3+2gx_3+2fy_3+c=0 \ldots. (2).$$

What line does the equation

$$axx_1+h(x_1y+xy_1)+byy_1+2gx_1+2fy_1+c=0 \ldots. (3)$$

represent? Subtracting (2) from (1), (3) from (1), and (2) from (3), we find

$$(ax+hy+g)+(hx+by+f)\left(\frac{y_3-y_2}{x_3-x_2}\right)=0 \ldots. (4),$$

$$(ax+hy+g)+(hx+by+f)\left(\frac{y_1-y_2}{x_1-x_2}\right)=0 \ldots. (5),$$

$$(ax+hy+g)+(hx+by+f)\left(\frac{y_3-y_1}{x_3-x_1}\right)=0 \ldots. (6).$$

Since (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear,

$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_1-y_3}{x_1-x_3}=\frac{y_3-y_2}{x_3-x_2}=m, \text{ say.}$$

Therefore (4), (5) and (6) are identical. That is, the same straight line $(ax+hy+g)+(hx+by+f)m=0$ passes through all three vertices of a triangle, which can only happen when the three sides are concurrent. Hence (3) passes through the intersection of (1) and (2).

ROLLA, MO.

COMPUTATION OF LOGARITHMS.

By A. W. WILLIAMSON, Augustana College, Illinois.

A table of logarithms may be computed more expeditiously by using the original series than by the usual method.

$$\text{I.}\quad l\tfrac{5}{4}=-l\tfrac{4}{5}=\tfrac{1}{5}+\frac{1}{2.5^2}+\frac{1}{3.5^3}+\ldots.(1),$$

$$l\tfrac{6}{5}=l(1+\tfrac{1}{5})=\tfrac{1}{5}-\frac{1}{2.5^2}+\frac{1}{3.5^3}-\ldots.(2).$$

Sum (2) by taking twice the sum of its negative terms from $l\frac{5}{4}$. Find in like manner the logarithms of $\frac{9}{10}$, $\frac{11}{10}$, $\frac{49}{50}$, and $\frac{51}{50}$. Then $l\frac{3}{2}=l\frac{6}{5}-l\frac{4}{5}$, $l\frac{3}{5}=l\frac{9}{10}-l\frac{3}{2}$, $l2=l\frac{6}{5}-l\frac{3}{5}$, $l3=l(3\div2)+l2$, $l5=l(5\div3)+l3$, $l7=\frac{1}{2}l49$, $l11=l(11\div10)+l10$, $l13=l(1000+1)-l77$, $l17=l51-l3$, $l19=l(400-1)-l21$, $l23=l(300-1)-l13$, $l29=l(2000+1)-l69$.

II. The following sets of series converge more rapidly. Find the logarithms of $\frac{9}{10}$, $\frac{11}{10}$, $\frac{19}{20}$, $\frac{21}{20}$, $\frac{24}{25}$, $\frac{26}{25}$, $\frac{49}{50}$, and $\frac{51}{50}$ as above. Then $l\frac{441}{400}=2l\frac{21}{20}$, $l\frac{9}{8}=l\frac{441}{400}-l\frac{49}{50}$, $l\frac{5}{4}=l\frac{10}{9}+l\frac{9}{8}$, $l\frac{25}{16}=2l\frac{5}{4}$, $l\frac{3}{2}=l\frac{25}{16}+l\frac{24}{25}$, $l\frac{9}{4}=2l\frac{3}{2}$, $l2=l\frac{9}{4}-l\frac{9}{8}$, $l3=l\frac{3}{2}+l2$, $l5=l\frac{5}{4}+l4$, etc.

III. Divide 1 by 2, the quotient by 2, and so on; then divide the first term by 1, the second by 2, etc., placing the divisors in a second column and the quotients in a third. The sum of the third column is evidently the logarithm of 2. Subtracting twice the sum of the even terms of the third column from its sum gives the logarithm of 3. The logarithms of all other numbers up to 100 may be found by dividing terms of the first column including unity only by powers of 10 and the number of the term. In computing an entire table this method would prove more expeditious than either of the preceding.

$l5=l(2^4-1)-l3$, $l7=l(2^6-1)-l9$, $l11=l(100-1)-l9$, $l13$, $l19$, and $l29$ as above; $l17=l(2^8-1)-l15$, $l23=l(160+1)-l7$, $l31=l(2^{10}-1)-l33$, $l37=l(1000-1)-l27$, $l41=l(2^{10}+1)-l25$, $l43=l(4000-1)-l93$, $l47=l(800-1)-l17$, $l53=l(160-1)-l3$, $l59=l(100\times2^{15}+1)-l(3^2.11^2.17)$, $l61=l(1280+1)-l21$, $l67=l201-l3$, $l71=l(640-1)-l9$, $l73=l(2^9-1)-l7$, $l79=l(100\times2^6-1)-l81$, $l89=l(800+1)-l9$, $l97=l(100\times2^5+1)-l33$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

165. Proposed by B. F. FINKEL. A.M., M.Sc., Professor of Mathematics and Physics, Drury College. Springfield, Mo.

A borrows \$2000 and agrees to pay back principal and interest in 100 equal monthly payments. Find the monthly payment. What would he have to pay yearly on the same conditions in order to discharge the debt in 100 months?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. E. SANDERS, Hackney, O.; M. E. GRABER, Heidelberg University, Tiffin, O.; and G. W. GREENWOOD, A. B., McKendree College, Lebanon, Ill.

As no rate of interest is named we will use 6%, and solve the second part first. 100 months$=8\frac{1}{3}$ years. Let p=principal, r=rate, x=yearly payment. Then $p(1+r)-x$=what remains after first payment; $p(1+r)^2-x(1+r)-x$= what remains after second payment.

$\therefore p(1+r)^8-x(1+r)^7-x(1+r)^6-\ldots.-x(1+r^2)^2-x(1+r)-x$=what remains after the eighth payment.

$x[1+(1+r)+(1+r)^2+\ldots.+(1+r)^6+(1+r)^7]=[x(1+r)^8-x]/r.$

$\therefore p(1+r)^8(1+\frac{1}{3}r)-x/r[1+r)^8-1](1+\frac{1}{3}r)-\frac{1}{3}x=0.$

$$\therefore x=\frac{pr(1+r)^8(3+r)}{[(1+r)^8-1](3+r)-r}=\frac{120\times1.5938481\times3.06}{.5938481\times3.06-.06}=\$333.069.$$

For monthly payments,

$$x=\frac{pr(1+r)^{100}}{(1+r)^{100}-1}=\frac{2000\times.005(1.005)^{100}}{(1.005)^{100}-1}=\frac{10\times1.64666849}{.64666849}=\$25.464.$$

ALGEBRA.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If n, $n+2$, $n+6$, $n+8$, $n+12$ are all primes, find the form of n.

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Every prime number ends in 1, 3, 7, or 9. Hence, n cannot end in 3, 7, or 9; for, if n ended in 3, $n+2$ would end in 5 and thus be composite; if n ended in 7, $n+8$ would end in 5; if n ended in 9, $n+6$ would end in 5. Hence, n must end in 1, for n greater than 10. Hence, n is of the form $10m+1$. The only value of n less than 10 is 5, the other primes being 7, 11, 13, 17, and 19.

If $m=10$, $n=11$; $n+2=13$, $n+6=17$, $n+8=19$, $n+12=23$.

If $m=100$, $n=101$; if $m=148$, $n=1481$; if $m=1942$, $n=19421$; if $m=2101$, $n=21011$; if $m=2227$, $n=22271$; if $m=4378$, $n=43781$, etc.

169. Proposed by **JOHN M. COLAW, A. M.,** Monterey, Va.

Solve $x^2+y+z=a\ldots.(1).$

$x+y^2+z=b\ldots.(2),$

$x+y+z^2=c\ldots.(3).$

I. Solution by **L. C. WALKER, A. M.,** Graduate Student, Leland Stanford Jr. University, Cal.

Eliminating z between (1) and (2); x between (2) and (3); y between (3) and (1), we obtain

$$(x-y)(x+y-1)=a-b\ldots.(4),$$
$$(y-z)(y+z-1)=b-c\ldots.(5), \text{ and}$$
$$(z-x)(z+x-1)=c-a\ldots.(6).$$

By addition of corresponding members of (4), (5), and (6), we have

$$(x-y)(x+y-1)+(y-z)(y+z-1)+(z-x)(z+x-1)=0\ldots.(7).$$

It is easy to see that either $x=y=z\ldots.(8)$, or $x+y-1=y+z-1=z+x-1=0\ldots.(9)$ will satisfy (7).

Now from (8) and (1); from (8), (2); from (8), (3); from $y+z=1$, (1); from $z+x=1$, (2); and from $x+y=1$, (3); we obtain $x=y=z=-1\pm\sqrt{(a+1)}$, $-1\pm\sqrt{(b+1)}$, or $-1\pm\sqrt{(c+1)}$, and

$$x=\pm\sqrt{(a-1)},$$
$$y=\pm\sqrt{(b-1)},$$
$$z=\pm\sqrt{(c-1)}.$$

II. Solution by **G. B. M. ZERR, A.M., Ph.D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x+y+z=s$. Then

$$x^2+s-x=a, \text{ or } x=\tfrac{1}{2}\pm\sqrt{[a-s+\tfrac{1}{4}]}\ldots.(1),$$
$$y^2+s-y=b, \text{ or } y=\tfrac{1}{2}\pm\sqrt{[b-s+\tfrac{1}{4}]}\ldots.(2),$$
$$z^2+s-z=c, \text{ or } z=\tfrac{1}{2}\pm\sqrt{[c-s+\tfrac{1}{4}]}\ldots.(3).$$

(1)+(2)+(3) gives $s=\frac{3}{2}\pm\{\sqrt{[a-s+\frac{1}{4}]}+\sqrt{[b-s+\frac{1}{4}]}+\sqrt{[c-s+\frac{1}{4}]}\}$.

Let $s-\frac{1}{4}=m\ldots.(4)$.

Then $m-\frac{5}{4}\mp\sqrt{[a-m]}=\pm\{\sqrt{[b-m]}+\sqrt{[c-m]}\}\ldots.(5)$.

Squaring (5),

$$m^2-\tfrac{3}{2}m+\tfrac{25}{16}+a-b-c=2\sqrt{\{[b-m][c-m]\}}\pm2(m-\tfrac{5}{4})\sqrt{[a-m]}\ldots.(6).$$

Squaring (6),

$$m^4+m^3-2(a+b+c+\tfrac{69}{16})m^2+7(a+b+c+\tfrac{25}{112})m+a^2+b^2+c^2+\tfrac{625}{256}$$
$$-2(ab+ac+bc)-\tfrac{25}{8}(a+b+c)=\pm8(m-\tfrac{5}{4})\sqrt{\{[a-m][b-m][c-m]\}}$$

or $m^4+m^3-Am^2+Bm+C=\pm8(m-\frac{5}{4})\sqrt{\{[a-m][b-m][c-m]\}}$, suppose.

Squaring this last equation

$$m^8+2m^7+(1-2A)m^6+(64+2B-2A)m^5+[160-64(a+b+c)+A^2$$
$$+2C+2B]m^4+[64(ab+ac+bc)-160(a+b+c)+100+2C-2B]m^3$$
$$+[160(ab+ac+bc)-100(a+b+c)-64abc+B^2-2AC]m^2$$
$$+[100(ab+ac+bc)+160abc+2BC]m+C^2-100abc=0.$$

This equation gives m. m in (4) gives s, and s in (1), (2), (3) gives x, y, z.

Also solved by *MARCUS BAKER*, who combines 169 and 173 under one solution.

170. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

Find by strictly quadratic methods at least one set of values of x and y in the equations $x^2y^2+x=a=38$, and $xy+y^2=b=15$.

Comments and Analysis by MARCUS BAKER. Washington. D. C.

The values of xy and of x from the second equation substituted in the first one give an equation in y; and similarly the values of xy and of y^2 from the second equation substituted in the first one give an equation in x. The two equations resulting from this elimination are

$$x^5+(b^2-a)x^4+2bx^3+(1-2ab)x^2-2ax+a^2=0\ldots.(1),$$
$$y^5\qquad\qquad -2by^3-\qquad\qquad y^2+(b^2-a)+b=0\ldots.(2),$$

equations of the fifth degree and of course insoluble by quadratics. Restoring numbers, the equations become

$$x^5+187x^4+30x^3-1139x^2-76x+1444=0\ldots.(3).$$
$$y^5\qquad\qquad -30y^3-\qquad y^2+187y+\ 15=0\ldots.(4).$$

It is obvious by inspection of the original equations that $x=2$ and $y=3$ are a pair of roots; therefore (3) is exactly divisible by $x-2$ and (4) is similarly divisible by $y-3$. Performing this division there results

$$x^4+189x^3+408x^2-323x-722=0\ldots.(5),$$
$$y^3+\ \ 3y^3-\ \ 21y^2-\ \ 64y-\ \ \ 5=0\ldots.(6),$$

equations of the fourth degree. Can these equations be solved by *strictly quadratic methods?* The answer depends on the meaning of the italicized words. Every quartic is solvable after the manner of quadratics in terms of an auxiliary involved in a cubic equation; and every cubic is solvable after the manner of quadratics in terms of an auxiliary involved in a quadratic equation; and every quadratic is solvable in terms of an auxiliary involved in a simple equation. In a certain sense, therefore, *every* biquadratic is solvable by quadratics. If by *strictly quadratic methods* it is implied that the auxiliary involved in the cubic is

excluded it only remains to apply the test for determining whether in the proposed equations (5) and (6), the auxiliary cubic is necessary. The general quartic

$$x^4+ax^3+bx^2+cx+d=0$$

is solvable by quadratics, without the intervention of a cubic, as shown by Thomas Simpson, in his Algebra, in three cases, viz., when

$$c=\tfrac{1}{2}af, \text{ or } a\sqrt{d} \text{ or } 2\sqrt{df},$$

where, for brevity, $f=b-\frac{1}{4}a^2$. Considering equation (6),

$$\begin{array}{ll} a=+3 & d=-4 \\ b=-21 & f=-23\frac{1}{4} \\ c=-64 & \end{array}$$

whence $\frac{1}{2}af=-34\frac{7}{8}$; $a\sqrt{d}=3\sqrt{-5}$; $2\sqrt{df}=5\sqrt{31}$.

As none of these quantities equal c, the conclusion is that y, and similarly x, cannot be determined by *strictly quadratic methods*.

Also solved by *J. E. SANDERS*, and *G. B. M. ZERR*.

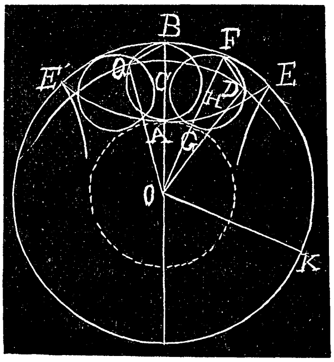
GEOMETRY.

191. Proposed by J. V. ADAMS, St. Louis. Mo.

Trisect any angle by means of the hypocycloid.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelyhia, Pa.; JOHN J. QUINN, Warren High School, Warren, Pa.; and M. E. GRABER, Heidelberg University, Tiffin, O.

Let EAE' be an arc of the hypocycloid from cusp to cusp; R=radius of fixed circle; r=radius of generating circle; AQB the central generating circle; C its center; Q any point on this circle. Join QB, QO. With QO as a radius and O as a center describe the arc QD meeting the hypocycloid in D. Let GDF be the generating circle when D is the generating point, GHF its diameter. Draw DF; construct angle BOK=to angle ACQ. Now arc GD=arc AQ, arc BQ=arc DF=arcFE.



∴ arc GD=arc BF=arc AQ. Arc BF measures angle BOF, arc AQ measures angle ACQ. But arc $BF=r/R$ arcBEK.

∴ Angle $BOF=r/R$ angle $BOK=r/R$ angle ACQ. If $R=3r$, angle $BOF=\frac{1}{3}$angle ACQ.

Therefore, by means of a suitable hypocycloid any angle may be divided in any given ratio. This property is applicable to the epicycloid also.

Also solved by the *PROPOSER*.

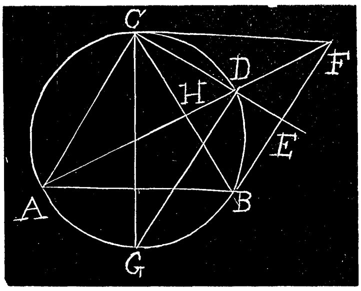
192. Proposed by ALFRED HUME, C. E., D. Ss., Professor of Mathematics, University of Mississippi, University, Miss.

Of all triangles with a common base and inscribed in the same circle, the isosceles is the maximum and has the maximum perimeter. Prove geometrically.

Solution by G. B. M. ZERR, A.M., Ph.D., The Temple College. Philadelphia, Pa., and JEANNETTE BROOKS, S. B., Chicago, Ill.

Let ABC be the isosceles triangle and ADB any other triangle, $AD>DB$. Produce AD to F, making $DF=DB$. Draw CF, BF, CD, DG. Then $\angle ADC= \angle FDE$; $\angle ADC$ is measured by $\frac{1}{2}(\text{arc}AC)=\frac{1}{2}(\text{arc}CDB)=\angle BDE$.

$\therefore \angle ADC=\angle BDE=\angle FDE$.



$\therefore CDE$ is perpendicular to BF at its mid-point. $\therefore CB=CF$.

Now $AC+CF>AF$. But $AF=AD+DB$, and $AC+CF=AC+CB$.

$\therefore AC+CB>AD+DB$. $\therefore AC+CB+AB>AD+DB+AB$.

$\dfrac{\triangle ADC}{\triangle BDC}=\dfrac{AC\times AD}{BC\times BD}$, but $AC=BC$ and $AD>BD$.

$\therefore \triangle ADC>\triangle BCD$. Take away the common triangle CHD and we get $\triangle AHC=\triangle BHD$.

$\therefore \triangle ABH+\triangle AHC>\triangle ABH+\triangle BHD$. $\therefore \triangle ABC>\triangle ABD$.

Also solved by *CLARENCE A. SHORT*, and *LON C. WALKER*.

CALCULUS.

158. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

It is required to cut a hole a inches square, for a crank shaft, through the center of a grindstone b inches thick at the outer edge, c inches thick at the center, and d inches in diameter. How many cubic inches will have to be cut out?

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Ohio University, Athens, O., and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

The equation to a right circular cone, its base being the plane of x, y, and center the origin of coördinates, is $x^2+y^2=\dfrac{r^2}{h^2}(h-z)^2$; r, the radius of base, and h, the altitude. In this example, $h=\frac{1}{2}c$, and $r=\dfrac{cd}{2(c-b)}$. The required volume $=\int\int\int dxdydz$. The limits of z are $\dfrac{c}{2}\Big(1-\dfrac{1}{r}\sqrt{(x^2+y^2)}\Big)$, $-\dfrac{c}{2}\Big(1-\dfrac{1}{r}\sqrt{(x^2+y^2)}\Big)$; of y, $\frac{1}{2}a$, $-\frac{1}{2}a$; of x, $\frac{1}{2}a$, $-\frac{1}{2}a$.

$$\therefore \int\int\int dxdydz=c\int\int dxdy-\frac{c}{r}\int\int dxdy\sqrt{(x^2+y^2)}=ac\int dx$$

$$-\frac{ac}{2r}\int dx\sqrt{(x^2+\tfrac{1}{4}a^2)}-\frac{c}{r}\int x^2dx\,\log\left(\frac{c+\sqrt{(x^2+c^2)}}{2x}\right)$$

$$=a^2c\left(1-\frac{a}{6r}[\sqrt{2}+\log(1+\sqrt{2})]\right).$$

Also solved by the *PROPOSER*.

II. Solution by G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Take the center of the grindstone as origin, Oz being the axis, and Ox, Oy being perpendicular to the sides of the opening. Using z, r, θ coördinates, the limits will be, respectively,

$$\frac{cd-2r(c-b)}{2d},\quad \frac{a}{2\cos\theta},\quad \tfrac{1}{4}\pi.$$

$$V=16\int\int\int rdzdrd\theta=\frac{8}{d}\int\int[cdr-2r^2(c-b)]drd\theta=ca^2\int\frac{d\theta}{\cos^2\theta}$$

$$-\frac{2a^3(c-b)}{3d}\int\frac{d\theta}{\cos^3\theta}=ca^2-\frac{a^3(c-b)}{3d}[\sqrt{2}-\log\tan\tfrac{1}{8}\pi].$$

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Solve $\dfrac{d^2u}{dx^2}=\dfrac{1}{m}\left(\dfrac{du}{dt}\right)$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

This is solved in Forsyth's *Differemtial Equations*, Edition of 1885, Art. 259, where a^2 is the "m" of this problem, and is by Poisson's extension of a method due originally to Laplace. In the article referred to we have given

$$u=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-u^2}f(x+2uat^{\frac{1}{2}})du,$$

with "another form may be given to this result by substituting λ for $x+2uat^{\frac{1}{2}}$. Then u becomes

$$\frac{1}{2a(\pi t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-[(x-\lambda)^2/4a^2t]}f(\lambda)d\lambda.$$

Now $f(\lambda)$ is an arbitrary function; if we choose to assume its value to be zero everywhere except when $\lambda=r$, and then write $f(\lambda)d\lambda=H$, we have

$$u=\frac{H}{2a(\pi t)^{\frac{1}{2}}}e^{-[(x-r)^2/4a^2t]}."$$

II. Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

The equation may be written

$$\left[\frac{d}{dt}-m\left(\frac{d}{dx}\right)^2\right]u=0;\text{ from which } u=\left(\frac{d}{dt}-m\frac{d^2}{dx^2}\right)^{-1}0.$$

By integrating with respect to t, we get

$$u=e^{mt(d^2/dx^2)}\phi(x)=\phi(x)+mt\frac{d^2\phi(x)}{dx^2}+\frac{m^2t^2}{1.2}\frac{d^4\phi(x)}{dx^4}+\ldots\text{ etc.}$$

By integrating with respect to x we shall have two arbitrary functions of t, since the differential with respect to x is of the second order. Now writing the equation in the form

$$\left[\frac{d}{dx}-\frac{1}{m^{\frac{1}{2}}}\left(\frac{d}{dt}\right)^{\frac{1}{2}}\right]\left[\frac{d}{dx}+\frac{1}{m^{\frac{1}{2}}}\left(\frac{d}{dt}\right)^{\frac{1}{2}}\right]u=0,$$

we obtain $u=e^{\sqrt{[(1/m)(d/dt)]}x}\phi(t)+\varepsilon^{-\sqrt{[(1/m)(d/dt)]}x}\psi(t)$

$$=F(t)+\frac{1}{m}\frac{x^2}{1.2.3}\frac{dF(t)}{dt}+\frac{1}{m^2}\frac{x^4}{1.2.3.4}\frac{d^2F(t)}{dt^2}+\ldots\text{ etc.}$$

$$+xf(t)+\frac{1}{m}\frac{x^3}{1.2.3}\frac{df(t)}{dt}+\frac{1}{m^2}\frac{x^5}{1.2.3.4.5}\frac{d^2f(t)}{dt^2}+\ldots\text{ etc.},$$

where $\phi(t)+\psi(t)=F(t)$, and $(d/dt)^{\frac{1}{2}}[\phi(t)-\psi(t)]=f(t)$.

This is the equation for determining the linear transformation of *heat* in an infinite solid.

Also solved by *G. B. M ZERR*, and *M. E. GRABER*.

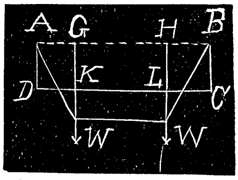
MECHANICS.

149. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From two points in the same horizontal line hangs a light inextensible string, on which are threaded two beads of equal mass. The beads start from rest in the position in which the terminal portions of the string are vertical and move symmetrically towards each other in the vertical plane. Find the path of each bead, and the tension of the string at any point in the path.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the points from which the weightless string hangs; D, C the position of the beads when the string is stretched and the beads are on the point of starting so that AD, BC are vertical; let the mid-point of KL be the origin; E, F the position of the beads at any time; $2a$=length of string; $2b=AB$; W=



weight of each bead; (x, y) the coördinates of F. Then we have $\sqrt{(HB^2+HF^2)}+\frac{1}{2}EF=a$.

$\therefore \sqrt{[(b-x)^2+(a-b+y)^2]}+x=a$,

or $y^2+2(a-b)y+2(a-b)x=2b(a-b)$,

or $(y+a-b)^2+2(a-b)x=(a-b)(a+b)=a^2-b^2$.

$\therefore (y+a-b)^2=(a-b)(a+b-2x)$, this is the locus of the bead F. $\therefore$ Each bead describes the arc of a parabola. Let T=tension of the string at any point, $\theta=\angle HFB$; resolving vertically, $\frac{1}{2}T\cos\theta=W$, or $T=2W/\cos\theta$. Now $1/\cos\theta=BF/HF=\sqrt{[(b-x)^2+(a-b+y)^2]}/(a-b+y)$.

$\therefore 1/\cos\theta=(a-x)/(a-b+y)$. $\therefore T=2W(a-x)/(a-b+y)$.

When $x=b$, $y=0$, and $T=2W$.

$\therefore x=0$, $y=\sqrt{(a^2-b^2)}-(a-b)$, $T=2aW/\sqrt{(a^2-b^2)}$.

DIOPHANTINE ANALYSIS.

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104. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

(1). The cube root of three cube numbers equals the square root of two square numbers. Determine the numbers.

(2). The sum of the square roots of three square numbers equals the sum of the cube roots of three cube numbers. Determine the numbers.

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

If we assume the three cube numbers to be x^3, y^3, and z^3, and the two square numbers to be u^2 and v^2, we are to find values of x, y, z, u, and v so that the equation $x+y+z=u+v$ shall be satisfied. Any four of these values may be selected at pleasure and the fifth one may then be determined. For example, let $z=1$, $y=2$, $u=3$, $v=4$. Then $x=4$. The cube numbers are 64, 8, 1, and the two square numbers are 9 and 16. By assuming any values whatever for y and z, and any values of u and v such that their sum is greater than the sum of y and z, as many positive numbers may be found satisfying the conditions of the problem as may be desired. The second part of the problem may be solved in the same way.

105. Proposed by HARRY S. VANDIVER, Bala, Pa.

Every odd factor of a^n+b^n is of the form $1(\mathrm{mod}\,2n)$.

No solution of this problem has been received.

106. Proposed by L. C. WALKER, A. M., Graduate Student. Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5 the first triangle, find the sides of the next five triangles, and a general expression for the sides of the nth triangle.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; CHAS. C. CROSS, Memphis, Tenn.; and J. E. SANDERS, Hackney, Ohio.

Let $x-1$, x, and $x+1$=the sides of a rational triangle.

Then, $(Area)^2=\dfrac{3x}{2}\cdot\dfrac{x}{2}\cdot\dfrac{x+2}{2}\cdot\dfrac{x-2}{2}=\dfrac{3x^2(x^2-4)}{16}=\square$.

$\therefore$ $3(x^2-4)=\square=9m^2$; and $x=\sqrt{(3m^2+4)}$.

To rationalize $\sqrt{(3m^2+4)}$, we use the convergents of the $\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}$, etc. Beginning with $\frac{2}{1}$ and taking the alternate convergents, we put x=twice the numerators and m=twice the denominators, and obtain the following set of pairs of values:

x=4, 14, 52, 194, 724, 2702, etc.
m=2, 8, 30, 112, 418, 1560, etc.

$\therefore$ The sides of the first six triangles are 3, 4, 5; 13, 14, 15; 51, 52, 53; 193, 194, 195; 723, 724, 725; 2701, 2702, 2703. The area=$3mx/4$.

Taking x_{n-2} and x_{n-1} as any two consecutive middle sides, we find the next middle side, or $x_n=4x_{n-1}-x_{n-2}$. This law of formation gives the following general expression for the middle side of the nth triangle, using 4 and 2 (=middle side of "straight line" triangle) as the middle sides of two consecutive triangles:

$$x_n=4\Big[4^{n-1}-(n-2)4^{n-3}+\frac{(n-3)(n-4)}{2}4^{n-5}-\frac{(n-4)(n-5)(n-6)}{2\times3}4^{n-7}+$$

$$\frac{(n-5)(n-6)(n-7)(n-8)}{2\times3\times4}4^{n-9}-\ldots\Big]-2\Big[4^{n-2}-(n-3)4^{n-4}+\frac{(n-4)(n-5)}{2}4^{n-6}$$

$$-\frac{(n-5)(n-6)(n-7)}{2\times3}4^{n-8}+\frac{(n-6)(n-7)(n-8)(n-9)}{2\times3\times4}4^{n-10}-\ldots\Big].$$

In finding values, we use those terms only which, in the substitutions for n, have *positive* and *zero* exponents.

Take $n=5$; then $x_5=4[4^4-3\times4^2+4^0]-2[4^3-2\times4']=724$.

Excellent solutions were received from *G. B. M. ZERR*, *J. SCHEFFER*, and the late *J. H. DRUMMOND*.

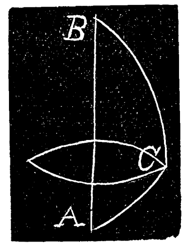
AVERAGE AND PROBABILITY.

128. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Two small circles are drawn on the surface of a sphere so as to intersect; find average area of the spherical triangle formed by joining the poles and one of the intersections of the small circles with arcs of great circles.

Solution by the PROPOSER.

Let A and B be the poles of the two small circles; C their intersection; join AC, BC, AB by arcs of great circles. Giving AB, AC, BC all the values within the limits of the problem, we are required to find the average area of ABC; and we may consider one of the poles, as A, to be fixed. Let r=radius of sphere, arc $AC=rx$, $BC=ry$, $AB=rz$, $\angle CAB=\theta$, $\angle CBA=\phi$, $\angle ACB=\psi$, area $ABC=r^2(\theta+\phi+\psi-\pi)=u$.



Then $\cos z=\cos x\cos y+\sin x\sin y\cos\psi$....(1).

$\sin z\cos\theta=\sin x\cos y-\cos x\sin y\cos\psi$....(2).

$\sin z\cos\phi=\cos x\sin y-\sin x\cos y\cos\psi$....(3).

An element of surface at B is $2\pi r^2\sin z dz$. The limits of x are 0 and $\frac{1}{2}\pi$; from $y=0$ to $y=x$, the limits of z are $x-y$ and $x+y$; and from $y=x$ to $y=\frac{1}{2}\pi$, the limits of z are $y-x$ and $y+x$. Hence the required average surface is

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}urdy.2\pi r^2\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{y+x}urdy.2\pi r^2\sin zdz\right]rdx}{\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}rdy.2\pi r^2\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{x+y}rdy.2\pi r^2\sin zdz\right]rdx}$$

$$=\frac{r^2}{2}\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}(\theta+\phi+\psi-\pi)dy\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{x+y}(\phi+\theta+\psi-\pi)dy\sin zdz\right]dx$$

$$\int(\theta+\phi+\psi-\pi)\sin zdz=-2\theta\cos^2\tfrac{1}{2}z-2\phi\cos^2\tfrac{1}{2}z-\psi\cos z+\pi\cos z+2\int\cos^2\tfrac{1}{2}zd\theta$$

$$+2\int\cos^2\tfrac{1}{2}zd\phi+\int\cos zd\psi.$$

Now $d\theta=-\sin y\cos\phi\operatorname{cosec} zd\psi$....(4), $d\phi=-\sin x\cos\theta\operatorname{cosec} zd\psi$....(5).

From (1) and (3), $\sin^2\frac{1}{2}z=\sin^2\frac{1}{2}(x+y)-\sin x\sin y\cos^2\frac{1}{2}\psi$....(6).

$\sin z\cos\phi=\sin(x+y)-2\sin x\cos y\cos^2\frac{1}{2}\psi$....(7).

From (4), (6), and (7),

$$2\int\cos^2\tfrac{1}{2}zd\theta=-\int\frac{2\sin y\cos^2\frac{1}{2}z\sin z\cos\phi\, d\psi}{\sin^2 z}=-\int\frac{\sin y\sin z\cos\phi d\psi}{2\sin^2\frac{1}{2}z}$$

$$=-\tfrac{1}{2}\sin y\int\left(\frac{\sin(x+y)-2\sin x\cos y\cos^2\frac{1}{2}\psi}{\sin^2\frac{1}{2}(x+y)-\sin x\sin y\cos^2\frac{1}{2}\psi}\right)d\psi$$

$$=-\psi\cos y+2\tan^{-1}\left(\frac{\sin\frac{1}{2}(x+y)\tan\frac{1}{2}\psi}{\sin\frac{1}{2}(x-y)}\right).$$

Similarly, $2\int \cos^2\frac{1}{2}z d\varphi = -\psi\cos x + 2\tan^{-1}\left(\frac{\sin\frac{1}{2}(y+x)\tan\frac{1}{2}\psi}{\sin\frac{1}{2}(y-x)}\right)$.

$$\int \cos z d\psi = \int (\cos x\cos y + \sin x\sin y\cos\psi)d\psi = \psi\cos x\cos y + \sin x\sin y\sin\psi.$$

When $z=x+y$, $\theta=\varphi=0$, $\psi=\pi$; when $z=x-y$, $\theta=\psi=0$, $\theta=\pi$; when $z=y-x$, $\varphi=\psi=0$, $\theta=\pi$.

$$\therefore\ \triangle = \tfrac{1}{2}\pi r^2 \int_0^{\frac{1}{2}\pi}\left[\int_0^x (3+\cos x\cos y-\cos x-\cos y)dy \right.$$

$$\left. + \int_x^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dy\right]dx.$$

$$\therefore\ \triangle = \tfrac{1}{2}\pi r^2 \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dxdy$$

$$= \tfrac{1}{4}\pi r^2 \int_0^{\frac{1}{2}\pi}(3\pi+2\cos x-\pi\cos x-2)dx = \tfrac{1}{8}\pi r^2(3\pi^2+4-4\pi)$$

129. Proposed by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Pa.

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being banker.]

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The chance of throwing 7 or 11 is $\frac{2}{9}$; the chance of throwing 2, 3, or 12 is $\frac{1}{9}$; the chance of throwing 4, 5, 6, 8, 9, or 10 is $\frac{2}{3}$. If A throws 4 the first throw the chance of winning the second throw is $\frac{1}{12}\cdot\frac{2}{3}$; of winning the third throw is $\frac{1}{12}\cdot\frac{2}{3},[1-(\frac{1}{12}+\frac{1}{6})]=\frac{1}{12}\cdot\frac{2}{3}\cdot\frac{3}{4}$.

$\therefore$ A's chance of winning on 4 is $\frac{2}{9}+\frac{1}{12}\cdot\frac{2}{3}[1+\frac{3}{4}+(\frac{3}{4})^2+(\frac{3}{4})^3+\ldots.]=\frac{4}{9}$.

A's chance of winning on 5 is $\frac{2}{9}+\frac{1}{9}\cdot\frac{2}{3}[1+\frac{13}{18}+(\frac{13}{18})^2+(\frac{13}{18})^3+.,..]=\frac{22}{45}$.

A's chance of winning on 6 is $\frac{2}{9}+\frac{5}{36}\cdot\frac{2}{3}[1+\frac{25}{36}+(\frac{25}{36})^2+(\frac{25}{36})^3+\ldots.]=\frac{52}{99}$.

A's chance of winning on 8, 9, or 10 is the same as for 6, 5, or 4.

$\therefore$ A's chance$=\frac{1}{3}(\frac{4}{9}+\frac{22}{45}+\frac{52}{99})=\frac{722}{1485}$; B's chance$=1-\frac{722}{1485}=\frac{763}{1485}$.

If the wager is given their expectation follows at once.

Also solved, with different results, by *W. W. LANDIS.*

MISCELLANEOUS.

128. Proposed by J. E. SANDERS, Hackney, O.

The sides of a trapezium are $a=29$, $b=32$, $c=40$, and $d=36$. If c is opposite a, and the diagonals equal, what is the length of either diagonal?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $AD=a=29$, $BC=q=32$, $CD=c=40$, $DA=d=36$, $AC=BD=x$, $\angle DOA =\theta$. Project AD, BC, AC on BD.

$\therefore x=d\cos ADB+b\cos CBD+x\cos\theta$. Multiply through by $2x$ and write, $-\cos(ADB+CAD)$ for $\cos\theta$.

$\therefore 2x^2=2dx\cos ADB+2bx\cos CBK-2x^2\cos(ADB+CAK)$; $2dx\cos AKB= d^2+x^2-a^2$; $2bx\cos CBK=d^2+x^2-c^2=2dx\cos CAK$. Substituting,

$$2x^2=d^2+x^2-a^2+d^2+x^2-c^2-1/2d^2(d^2+x^2-a^2)(d^2+x^2-c^2)$$
$$+1/2d^2\sqrt{\{[4d^2x^2-(d^2+x^2-a^2)^2][4d^2x^2-(d^2+x^2-c^2)^2]\}}.$$

Reducing and collecting,

$$2x^6-(a^2+b^2+c^2+d^2)x^4+[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]x^2$$
$$+(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0.$$

Restoring numbers, $2x^6-4761x^4+317712x^2+2238016=0$.

$\therefore x=48.07$ nearly.

Also solved by *LON C. WALKER*, *J. SCHEFFER*, and *D. B. NORTHRUP*. Mr. Northrup's result agreed with Professor Zerr's and was obtained by the method of trial and error.

129. Proposed by J. SCHEFFER, A. M,, Hagerstown, Md.

How high above the surface of the earth must an observer be elevated at the latitude $\phi(=39^\circ\ 19')$, the declination of the sun being $\delta(=23^\circ\ 27')$, in order to see the sun at midnight?

Solution by the PROPOSER.

The sun will be seen at midnight when the tangent drawn from the point to the earth strikes the sun when on the meridian at midnight. Denoting the required height above the earth by h, the radius of the earth by R, the latitude of the place by ϕ, and the declination of the sun by δ, we easily find $\sin(\phi+\delta)=\frac{R}{R+h}$, whence $h=\frac{R[1-\sin(\phi+\delta)]}{\sin(\phi+\delta)}=\frac{2R\sin^2[45-\frac{1}{2}(\phi+\delta)]}{\sin(\phi+\delta)}$.

For $\phi=39^\circ\ 19'$, $\delta=23^\circ\ 27'$, we get $h=495$ miles, nearly.

Also solved, with slightly different results, by *G. B. M. ZERR*, *S. HART WRIGHT*, and *G. W. GREENWOOD*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

174. Proposed by HARRY S. VANDIVER, Bala, Pa.

If the quantity x be expressed in the form of a continued fraction P_n/Q_n denoting the $(n+1)$th convergent, with x_n the corresponding complete quotient, then $\dfrac{P_{n-(k+1)}-Q_{n-(k+1)}x}{P_n-Q_nx}=(-1)^{k+1}x_n\times x_{n-1}\dots x_{n-k}$.

175. Proposed by W. J. GREENSTREET, M.-A., Editor of The Mathematical Gazette, Stroud, England.

Find the conditions that $\dfrac{x}{m+3}+\dfrac{y}{m-1}+\dfrac{z}{m-z}=1$, where m may be a, b, or c.

176. Proposed by MARCUS BAKER. U. S. Geological Survey, Washington, D. C.

Solve $x^2+y^2+z=a\dots(1)$, $x+y^2+z^2=b\dots(2)$, $x^2+y+z^2=c\dots(3)$.

GEOMETRY.

197. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Two points P_1, Q_1 are on a generator of a hyperboloid, and P_2, Q_2 the corresponding points on a confocal hyperboloid. Prove $P_1Q_1=P_2Q_2$.

198. Proposed by JOHN J. QUINN, Professor of Mathematics, Warren High School. Warren, Pa.

Trisect an angle, (1) by means of the cissoid; (2) by means of the paraboloid.

199. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College. Oberlin, O.

Two verticies of a given triangle move along fixed right lines; find the locus of the third. [From Salmon's Conics, Sixth Edition, p. 208, ex. 10.]

CALCULUS.

163. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Can there be a plane curve the length of which varies *directly as the abscissa* and *inversely as the ordinate* of any point on the curve?

164. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $m^2+n^2=1, m^2\cos^2\theta+n^2\cos^2\varphi=A$, $a^2b^2\sin^2\theta(m^2+n^2\cos^2\varphi)+a^2c^2\cos^2\theta\cos^2\varphi+b^2c^2\sin^2\varphi(n^2+m^2\cos^2\theta)=B$, $\sqrt{(1-m^2\sin^2\theta)}=\triangle(\theta)$, $\sqrt{(1-n^2\sin^2\varphi)}=\triangle(\varphi)$, prove that $\displaystyle\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\triangle(\varphi)}=\frac{\pi}{6}(a^2b^2+a^2c^2+b^2c^2)$.

MECHANICS.

153. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

An equiangular polygon consisting of equal, freely jointed rods, is hung up from a vertex, A. The vertices adjacent to A are connected by a light rod of such a length that the polygon is still regular. Find the stress in the rod and the reactions at the vertices.

154. Proposed by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio.

Find the form of the curve in a vertical plane such that a heavy bar resting on its concave side and on a peg at a given point, (the origin), may be at rest in all positions.

DIOPHANTINE ANALYSIS.

113. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Find the four least integral numbers such that the difference of every two of them shall be a square number.

114. Proposed by J. E. SANDERS, Hackney, O.

Find the least integral values (if any) of a, b, and c that will make $2(a+b+c)\pm 2\sqrt{[12ab-3(a+b-c)^2]}$ a square number for either sign of the radical.

AVERAGE AND PROBABILITY.

140. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Obtain the average area of a triangle formed by a tangent to the four—cusped hypocycloid and the coordinate axes.

141. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Upon a circular table, radius r, a *variable* square plate is thrown at random. What is the probability that the plate will lie wholly on the table?

MISCELLANEOUS.

135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Find invariants of the *second, third,* and *sixth degrees* in the coefficients of a binary quartic.

136. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

(1) Solve (to five places) the equations, $\sin(x+\frac{1}{6}\pi)=10\sin x$, and $a\cos\phi\log\sin\phi=p$ where a is small and positive, and $\varphi=\alpha+\kappa$, where κ is very small and α is not very small; (2) If $a\theta=b\varphi$ where a is prime to b, and $\sin\theta=p$, $\sin\varphi=q$, how many values of q are there for each of p? (3) if $2x=\sin^{-1}x$, show there is only one positive value of x, and find it.

NOTES.

Dr. O. D. Kellogg has been appointed instructor in mathematics at Princeton University. F.

Sir George Gabriel Stokes, F. R. S., Lucasian Professor of Mathematics at Cambridge, died February 1st, aged 83. D.

Lawrence Sluter Benson, a tireless disseminator of unsound mathematical doctrines, died at Newark, N. J., January 27 F.

Mr. P. A. Smith has resigned as instructor in mathematics at the University of Illinois to accept a position in Hiroshina Higher Normal School of Japan. D.

The College Entrance Examination Board has appointed as examiners in mathematics for 1903, Professor Charlotte A. Scott, Professor W. H. Metzler, and Mr. J. H. French. F.

At the University of Texas, Dr. H. Y. Benedict has been advanced to be Associate Professor of Mathematics, and Dr. M. B. Porter of Yale University has accepted the call to the head of the department. D.

Mr. W. J. C. Miller, for many years mathematical editor of the *Educational Times*, of London, died on February 11, at Bristol, England, at the age of seventy-one and a half years. For a biography of Mr. Miller see Vol. 3 of the MONTHLY. F.

An announcement has been made of the death on January 31, of Dr. Norman Macleod Ferres, F. R. S., Gonville and Caius College, Cambridge, in his seventy-fourth year. In 1851 he was senior wrangler and Smith's prizeman. Among his treatises is one on trilinear coördinates and one on spherical harmonics. D.

BOOKS.

Problems in Astrophysics. By Agnes M. Clerke, Author of "A History of Astronomy During the Nineteenth Century," "The System of the Stars," and other works. 8vo. Cloth, xvi + 568 pages. Price, $6.00. London: Adam & Charles Black. New York: The Macmillan Co.

While the object of this interesting and valuable book is not so much to instruct as to suggest, yet it admirably sets forth not only many of the problems already solved in this field of research, but also points out to those who are engaged in research work in Astronomy the many problems yet unsolved.

The book is divided into two parts,—the first part dealing with "Problems in Solar Physics" and the second part with "Problems in Siderial Physics." In the first part, among other subjects treated, are the following: *The Chemistry of the Sun, the Corona, the Sun's Rotation, and the Solar Cycle.* Some of the subjects treated in the second part

are, *Helium Stars, Hydrogen Stars, Solar Stars, the Evolution of the Stars, Eclipsing Stars,* and *Dark Stars.*

In the first part, 14 topics are considered and in the second, 41.

The book is one that every astronomer and every one interested in astronomy will want to have in his library. F.

Arithmetic for High Schools, Academies, and Normal Schools. By Oscar Lynn Kelso, M. A., Professor of Mathematics, Indiana State Normal School. 8vo. Leather back, cloth sides. x+274 pages. Price, 60c. New York: The Macmillian Co.

In this book are treated the usual subjects of arithmetic, viz.: Numeration and Notation, Addition, Subtraction, Multiplication, Division, Factoring, Fractions, Involution, Evolution, Compound Numbers, Mensuration, Longitude and Time, Percentage, Interest, and Stocks and Bonds. The author begins his treatment of Arithmetic with a short discussion on the origin of numbers. This will be especially helpful to most teachers.

The book is well written and the subjects are presented in a simple and logical manner. This Arithmetic will be greatly appreciated by the progressive teacher. F.

Introduction to the Theory of Algebraic Equations. By Leonard Eugene Dickson, Ph. D., Assistant Professor of Mathematics in the University of Chicago. 8vo. Cloth, v+104 pages. Price, $1.25. New York: John Wiley & Sons.

In the preparation of this excellent little work, Dr. Dickson has placed the teacher and student of elementary mathematics under great obligation to him. He has here produced a work so simple in its treatment and yet sufficiently comprehensive to enable those who desire to gain an insight into that profound and far-reaching branch of modern mathematics, the Theory of Groups, to take up the subject with interest and profit.

The work begins with the solutions of the general Quadratic, Cubic, and Quartic Equations. The second chapter deals with Substitutions; the third, with Substitution Groups; the fourth, with the General Equation from the Group Standpoint; the fifth, the Algebraic Introduction to Galoi's Theory; the sixth, the Group of an Equation; the seventh, Solution by Means of Resolvent Equations; the eighth, Regular Cyclic Equations—Abelian Equations; the ninth, Criterion for Algebraic Solvability; the tenth, Metacyclic Equations—Galoisian Equations; and the eleventh, An Account of More Technical Results. In the Appendix, the Fundamental Theorem of Symmetric Functions is proved and also the Theorem: *If a rational, integral function of* $x_1, \ldots, x_n$, *with constant coefficients, equals zero, it is identically zero.*

We most heartily recommend this work to all who wish to know something about the group theory and its applications. F.

The Elements of Plane Analytic Geometry. By George R. Briggs. Seventh Edition, revised and enlarged by Maxime Bôcher, Assistent Professor in Harvard University. 12mo. Cloth, v+191 pages. New York: John Wiley & Sons.

The present edition justifies its aim to cover the ground of the ordinary first course in Analytic Geometry as given in our colleges and technical schools. The additions by Professor Bocher are chiefly an elementary account of poles and polars and a brief, but satisfactory treatment of the general equation of the second degree.

There is no index and only a sparse table of contents; this, coupled with a lack of headings and display type, make the matter of reference difficult. This may, however, prevent the mechanical use of reference formula and encourage the solution *ab initio* of the elementary problems. D.

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APPRECIATIVE REMARKS ON THE THEORY OF GROUPS.

By PROF. G. A. MILLER.

The teacher of mathematics is frequently called upon to make comments on various lines of mathematical thought. In doing so it is very desirable to be able to make use of remarks by prominent mathematicians. Some of these naturally exaggerate the relative importance of a particular subject and their true value can only be appreciated when compared with similar remarks on other mathematical developments. A number of such collections of appreciative remarks on modern subjects would doubtless prove very useful, especially to those who are in doubt as to which field to choose as an object of investigation.

In a recent German review of Professor Dickson's Linear Groups,* the reviewer aptly remarks that group theory is especially cultivated by the English speaking people. As there are very few subjects in pure mathematics in regard to which such a remark could be made by one so well informed as Professor Loewy, statements with reference to the relative importance of this subject may be of especial interest to the English speaking students of mathematics. It may be observed that most of the statements given below are due to mathematicians whose native language is not English.

An important feature of group theory is exhibited by the new *International Catalogue of Scientific Literature*. Three subjects are classed under "Fundamental Notions;" viz., Bases of Arithmetic, General Algebra, and The Theory of Groups. As some of the leading mathematicians of the world helped to arrange this cat-

* Loewy, *Archiv der Mathematik und Physik*, Vol. 4, 1903, page 333.

alogue, this classification is very significant. The three subjects with which group theory has most direct contact according to this catalogue are: Theory of Algebraic Equations, Automorphic Functions, and Differential Invariants.

That there are many other subjects of direct contact may be seen from the following quotations: "The theory of congruences bases itself substantially upon the fundamental concept of mathematics, which is already the foundation of Poinsot's method, the concept of group."* "A large part of the theory of numbers is only the theory of abelian groups."† "It might be said of the most important parts of recent geometry that one conception dominates everywhere; that is the conception of the group."‡ "The most important of all these view points is furnished by the theory of groups, which is really a creation of our century and has shown its dominating influence in nearly all parts of mathematics; not only in the recent theories but also far towards the foundation of the subject, so that this theory can no longer be omitted in the elementary text-books."§

"There are two things which have become especially important for the latest development of algebra; that is, on the one hand the ever more dominating theory of groups whose systematizing and clarifying influence can be felt everywhere, and then the deep penetration of number theory."‖ "The concepts, group and invariant, take each day a more preponderant place in mathematics and tend to dominate this entire science."¶ "The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution-groups, as he might now but for inherited habits."**

"In fine, the principal foundation of Euclid's demonstrations is really the existence of the group and its properties. Unquestionably he appeals to other axioms which it is more difficult to refer to the notion of group. An axiom of this kind is that which some geometers employ when they define a straight line as the shortest distance between two points. But it is precisely such axioms that Euclid enunciates. The others, which are more directly associated with the idea of displacement and with the idea of groups are the very ones which he implicitly admits and which he does not deem even necessary to enunciate. This is tantamount to saying that the former are the fruit of later experience, that the others were first assimilated by us, and that consequently the notion of group existed prior to all others."††

"Although we can not give here a complete exposition of the fundamental results of Sophus Lie in the theory of continuous transformation groups, yet it is indispensable that we make some general remarks on this notion of continuous

* Bachmann, *Die Elemente der Zahlentheorie*, Vol. 1, 1892, Preface.

† Frobenius, *Sitzungsberichte*, 1893, page 627.

‡ Maschke, THE AMERICAN MATHEMATICAL MONTHLY, Vol. 9, 1902, page 214.

§ Pund, *Algebra mit Einschluss dei elementaren Zahlentheorie*, 1899, Preface.

‖ Weber, *Lehrbuch der Algebra*, Vol. 1, 1898, Preface

¶ Lie, *Le Centenaire de l' Ecole Normale*, 1895, page 485.

** Newcomb, *Bulletin of the New York Mathematical Society*, Vol. 3, 1893, page 107.

†† Poincare, *The Monist*, Vol. 9, 1898, page 34. On page 41 of the same article Poincare says, "What we call geometry is nothing but the study of formal properties of a certain continuous group; so that we may say, space is a group."

groups which play such an important rôle in the science of our epoch."* "The group concept was employed in the preceding century (about 1770) simultaneously by Lagrange and Vandermonde, and since this time it occupies a prominent place in the theory of algebraic equations. In regard to this it is only necessary to refer to the name of Galois. Hence group theory has been regarded as a supplement of algebra. This however is incorrect. For the group concept extends far beyond this into almost all other parts of mathematics."†

"It was reserved for Galois to place the theory of equations on a definite foundation by showing that to each equation there corresponds a substitution group in which are exhibited its essential characteristics, and especially those which relate to its solution by other auxiliary equations."‡ "I should reproach myself for forgetting, even in so rapid a resumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge."§

"The theory of groups, which is making itself felt in nearly every part of higher mathematics, occupies the foremost place among the auxiliary theories which are employed in the most recent function theory."‖ "It need scarcely be added that some modern mathematicians seem to avoid group theory even where it would simplify the treatment of the subject in hand. This seems to be true, for instance, of Hilbert's Grundlagen der Geometrie."¶

LELAND STANFORD UNIVERSITY.

OUR SYMBOL FOR ZERO.

By DR. GEORGE BRUCE HALSTED.

At the Paris International Congress, the paper of my erudite friend of Japanese days, Professor Fiyisawa, attracted, I believe, apart from the great address of Hilbert, more favorable attention than any other.

In praising this paper, I ventured there to emphasize, that the appearance in Japan, before any communication with Europe, of a positional notation for number with precisely the symbol for zero which we now use, and which, as Professor Cajori says in the February number of the AMERICAN MATHEMATICAL MONTHLY, has been supposed of Hindu origin, raised the question for the future historian of mathematics of the relation or connection between these two indubitable, however widely separated, appearances of the same peculiar symbol.

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* Picard, *Traite d' analyse*, Vol. 3, 1896, page 492.

† Klein, *Einleitung in die hoehere Geometrie* II, 1893, page 3.

‡ Jordan, *Traite des substitutions*, 1870, Preface.

§ Darboux, *Comptes Rendus*, Vol. 128, 1899, page 528.

‖ Fricke und Klein, *Automorphe Functionen*, Vol. 1, 1897, page 1.

¶ Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 272.

Professor M. Cantor, who was presiding over the section, seemed perturbed, perhaps at the very idea of future historians of mathematics, and strenuously insisted that the idea of positional notation for number and the symbolization of zero should be accredited to the Babylonians. Of course we know that there are some fragmentary matters which point to the Babylonians using sixty as a number radix, but I have never learned that any Babylonian symbol for zero has been discovered, and the question to which I called attention remained untouched. But now, if we may accept the testimony of Y. Mikami, of Tokio, one of those questions has been answered. He says, "I have found very important relations between the mathematics of India and of China." But we know that Japanese hieroglyphic writing, like Japanese Buddhism, came through China.

Therefore the appearance of our symbol for zero in Hindustan and Jăpan is not one of the mysterious and seemingly inexplicable coincidences, such as the prehistoric appearance and importance of the cross in the pre-Columbian religions of Yucatan and Mexico. For here we find a water-way connecting Hindustan and Japan through China. But the source of the flow still remains undetermined.

Use of the abacus might have been what suggested the symbolization of zero. But Hindustan from time immemorial used systems of abacal calculation. Greece, and even mathematically stupid Rome, had the abacus.

Every claim for importance in the world of ideas hitherto made for China has evaporated into nothingness. They did not even invent gunpowder or the mariner's compass. Remember their exaltation and subsequent downfall in the history of astronomy.

So it will be as to their claim to our zero symbol. They got it, as they did their Buddhism, from India, and passed on both to Japan.

ANNAPOLIS, MD., *March 7, 1903.*

HARMONIC PAIRS IN THE COMPLEX PLANE.

A PURELY GEOMETRICAL TREATMENT FOR CERTAIN MAPS DEFINED BY THE SUBSTITUTION $w=\frac{1}{2}(z+\frac{1}{z})$.

By ARCHIBALD HENDERSON. Ph. D.. Associate Professor of Mathematics, University of North Carolina, Chapel Hill.

§1. A subject of especial interest in the Theory of Functions is the conception of harmonic pairs in the complex plane. The proof given below offers initially a direct interpretation of the equation

$$\left|\frac{z-z_1}{z-z_2}\right|=\rho\text{....(1)},$$

and leads subsequently to the interpretation of the harmonic relation

$$\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2'} = -1 \ldots. (2),$$

where z, z_1, z_2, z_1', z_2' are complex quantities and ρ is a constant.

In a well-known text-book* the method of interpreting equation (1) seems essentially artificial, although the logic is entirely sound. Instead of interpreting the equation directly, which is the problem set, the writer employs a property of the configuration, wholly unsuspected by one who reads the subject for the first time, which leads, after a series of transformations, to the equation in question.

For the following proof, several lemmas will be introduced.

Lemma 1. If $\frac{z-z_1}{z_2-z}=\kappa$, where κ is real, then z, z_1, and z_2 are collinear. Writing the equation in the form

$$z = \frac{z_1+\kappa z_2}{1+\kappa}$$

we may say that z divides the stroke from z_1 to z_2 in the ratio $+k : 1$.†

Lemma 2. If $\operatorname{am}\frac{z-z_1}{z-z_2}=a$, then z moves on the arc of a circle through the points z_1 and z_2.‡

Lemma 3. If $\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2}=\lambda$, where λ is a real quantity, then the four points z_1, z_2, z_1', and z_2' are concyclic.§

This lemma is an immediate consequence of Lemma 2.

Consider now the equation (1) above and its interpretation. It is satisfied *in particular* by the two equations

$$\frac{z-z_1}{z-z_2}=\rho, \quad \frac{z-z_1}{z-z_2}=-\rho,$$

shown by taking absolute values on both sides of these two equations. Hence the locus represented by equation (1) crosses the line through the two points z_1 and z_2 at the two points

$$z_\mu = \frac{z_1+\rho z_2}{1+\rho}; \quad z_\nu = \frac{z_1-\rho z_2}{1-\rho},$$

* Harkness and Morley's *Introduction to Analytic Functions*.
† Harkness and Morley: *Introduction to Analytic Functions*, §18.
‡ Ibidem, §21.
§ Burkhardt: *Functionen Theorie*, Erster Band, §15, VI.
Harkness and Morley: *Introduction to Analytic Functions*, §25.

which follows from Lemma 2. It is to be noted that since the points z_μ and z_ν divide the stroke from z_1 to z_2 both externally and internally in the same ratio, the pairs z_1 and z_2, z_μ and z_ν are harmonic.

Also $z_c = \frac{1}{2}(z_\mu + z_\nu)$

$$= \frac{1}{2}\left(\frac{z_1 + \rho z_2}{1+\rho} + \frac{z_1 - \rho z_2}{1-\rho}\right)$$

$$= \frac{z_1 - \rho^2 z_2}{1-\rho^2}$$

and this point divides the segment z_1 to z_2 externally in the ratio $1 : \rho^2$.

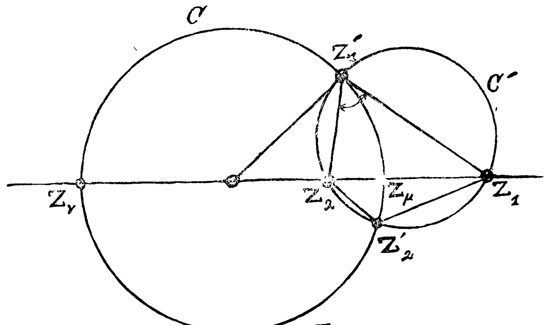


Fig. 1.

Now $\left|\frac{z - z_1}{z - z_2}\right| = \rho$ requires that $\frac{z - z_1}{z - z_2} = \rho(\cos\theta + i\sin\theta)$ and accordingly $z = \frac{z_1 - z_2\rho(\cos\theta + i\sin\theta)}{1 - \rho(\cos\theta + i\sin\theta)}$.

Forming now the difference $z - z_c$ we obtain after reduction

$$z - z_c = \frac{\rho(z_1 - z_2)}{1-\rho^2}(\cos\theta + i\sin\theta)\left[\frac{1-\rho(\cos\theta - i\sin\theta)}{1-\rho(\cos\theta + i\sin\theta)}\right].$$

Taking absolute values on both sides of this equation and noting that

$$|\cos\theta + i\sin\theta| = 1,$$

$$|1 - \rho(\cos\theta + i\sin\theta)| = |1 - \rho(\cos\theta - i\sin\theta)|$$

we obtain

$$|z - z_c| = \frac{\rho\,|z_1 - z_2|}{1-\rho^2} = \text{constant},$$

and since z_c is a fixed point, we have the final result, which may be stated as follows:

The equation $\left|\frac{z - z_1}{z - z_2}\right| = \rho$ represents a circle C of radius $\frac{\rho\,|z_1 - z_2|}{1-\rho^2}$ with its center on the line joining the points z_1 and z_2 and dividing the segment from z_1 to z_2 externally in the ratio $1 : \rho^2$.

Now through the points z_1 and z_2 draw any circle C', cutting the circle C in the two points z_1' and z_2'. The equations of the two finite portions of the arc of this circle C', by Lemma 2, are

$$\left.\begin{array}{l}\operatorname{am}\dfrac{z-z_1}{z-z_2}=\alpha \\ \\ \operatorname{am}\dfrac{z-z_1}{z-z_2}=\alpha-\pi\end{array}\right\}\ldots.(C').$$

That the circles C and C' are orthogonal can be shown in the following manner. We have the three equations

$$|z_1'-z_c|=\frac{\rho\,|z_1-z_2|}{1-\rho^2},$$

$$|z_1-z_c|=\left|z_1-\frac{z_1-\rho^2 z_2}{1-\rho^2}\right|=\frac{-\rho^2}{1-\rho^2}|z_1-z_2|,$$

$$|z_2-z_c|=\left|z_2-\frac{z_1-\rho^2 z_2}{1-\rho^2}\right|=\frac{-1}{1-\rho^2}|z_1-z_2|,$$

and therefore,

$$|z_1'-z_c|^2=|z_1-z_c|\times|z_2-z_c|.$$

That is, the join of the two points z_c and z_1' is a tangent to the circle and hence the circles C and C' are orthogonal.*

Since $\dfrac{|z_1'-z_c|}{|z_2-z_c|}=\dfrac{|z_1-z_c|}{|z_1'-z_c|}$, the triangles $z_c z_2 z_1'$ and $z_c z_1' z_1$ are similar, having the angle $z_2 z_c z_1'$ in common. Putting

$$|z_1-z_c|\times|z_2-z_c|=r^2,$$

the points z_1, z_2 are defined as *inverse* with respect to the circle of center z_c and radius r; and this circle will be said to be drawn *about* any such pair of points.

Another definition of inverse points, found in Harkness and Morley's *Introduction*, based on the orthogonality of the circles C and C', is as follows:

Two points z_1, z_2 are said to be inverse to a circle, when every circle through z_1, z_2 cuts the given circle orthogonally.

§2. The next problem is to interpret the geometric meaning of the equation

$$\frac{z_1'-z_1}{z_2-z_1'}\div\frac{z_2'-z_1}{z_2-z_2'}=-1.$$

Since -1 is a real quantity, the four points z_1, z_2, z_1', z_2' are concyclic (Lemma 3). Also clearing out and taking absolute values on both sides, we obtain

* The orthogonality of the two circles may also be found as follows: Isogonality is an invariant property from every linear substitution of the form $w=(az+b)/(cz+d)$. (Forsyth: *Theory of Functions*, §258, page 514). Making the substitution $w=(z-z_1)/(z-z_2)$ in the two equations $|(z-z_1)/(z-z_2)|=\rho$, $\operatorname{am}[(z-z_1)/(z-z^2)]=\alpha$, their maps are $|w|=\rho$, $w=\alpha$,—circle with the origin as center and a half-ray through the origin. Since these figures are orthogonal in the w-plane, their images in the z-plane are also orthogonal.

$$\frac{|z_1'-z_1|}{|z_1'-z_2|}=\frac{|z_2'-z_1|}{|z_2'-z_2|}$$

and hence the points z_1' and z_2' lie on the locus defined by the equation

$$\left|\frac{z-z_1}{z-z_2}\right|=\rho.$$

These four points z_1, z_2, z_1', and z_2' lie on the circle C', two of them, z_1' and z_2', are on C, while two of them, z_1 and z_2, are inverse with respect to C. The pairs z_1 and z_2, z_1' and z_2' are defined as *harmonic pairs* of points.

Since the straight line through z_1 and z_2 may be regarded as a circle of infinite radius, it follows that the pairs z_ν and z_μ, z_1 and z_2 are harmonic pairs on a line, a fact which was proved in a different manner at the beginning of §1.

II.

§3. A very suggestive problem, illustrative of the Riemann surface, is to find the images of certain configurations—straight lines through the origin and concentric circles with origin as center—defined by the equation of correspondence

$$w=\tfrac{1}{2}(z+\frac{1}{z})\ldots.(1).$$

The purely geometrical treatment given here is as simple and perhaps simpler than the analytical treatment.*

The equation indicates that the correspondence between the z and the w-planes is not (1, 1), since z takes two distinct values for one of w. The point w is constructed geometrically in the following manner: *Plot the points xepresented by z and $1/z$ and the mid-point of their join gives the point w.*

Consider a circle about the origin, defined by the equation $|z|=r$. If we write $w_1=z$, $w_2=1/z$, and consequently $w=\frac{1}{2}(w_1+w_2)$, we see that as w_1 describes the circle $|z|=r$ in the positive direction, w_2 describes *with the same same angular velocity* the circle $|z|=1/r$ in the opposite direction. This follows from the fact that if we denote

$$\left.\begin{aligned} z&=\rho(\cos\theta+i\sin\theta),\\ \frac{1}{z}&R(\cos\varphi+i\sin\varphi),\end{aligned}\right\}\ldots.(2),$$

the resulting conditions are

$$R=1/\rho,\quad \varphi=-\theta\ldots.(3).$$

* Holzmueller, *Isogonalen Verwandschaften*, §61, et seq.

If the two circles be described in the sense explained, the point w describes a certain curve which cuts the axes at the points $[\pm\frac{1}{2}(\frac{1}{r}+r), 0][0, \pm(\frac{1}{r}-r)]$ respectively. From the method of description it follows that the curve is symmetrical with respect to the axes. Describe about the origin as center a circle C of radius $\frac{1}{2}(1/r+r)$ and consider for a moment the point M, a point on the locus of the point w, derived from the points F and F' (Fig. 2). The line ML perpendicular to the X-axis is parallel to the line GF, and since ML is parallel to the base of the triangle FGF' and passes through the middle point M of one side FF' of the triangle, therefore it must pass through the middle point N of the other side OF'. Hence it meets the line OF' at the distance $\frac{1}{2}(1/r+r)$ from the center O, and accordingly the point N lies on the circle C. Now

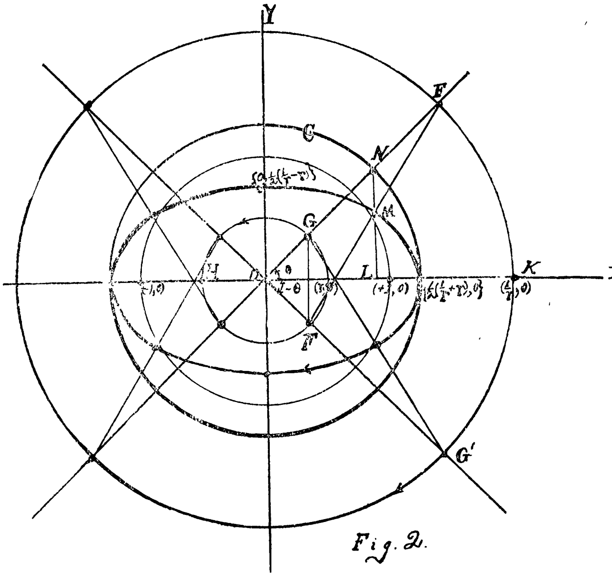


Fig. 2.

$$\frac{NM+ML}{NM}=\frac{NL}{\frac{1}{2}GF}=\frac{ON}{OG}=\frac{\frac{1}{2}(1/r+r)}{r}$$

from which we have

$$NM=\frac{r}{\frac{1}{2}(1/r-r)}ML$$

and substituting we obtain

$$\frac{NL}{ML}=\frac{\frac{1}{2}(1/r+r)}{\frac{1}{2}(1/r-r)}.$$

This proves that the point w describes an ellipse of semi-axes $\frac{1}{2}(1/r+r)$, $\frac{1}{2}(1/r-r)$.* If we have $0<r\overline{\gtrless}1$, the sense of description is opposite to that of the circle $|z|=r$. If, however, $r=1\ldots.\infty$, the ellipse and the circle $|z|=r$ are described in the same sense. Thus when G has reached the point $(0, r)$, G' has reached the point $(0, -\frac{1}{r})$ and M has reached the point $0, \frac{1}{2}(r-\frac{1}{r})$, a point below the X-axis; and so in other cases.

* C. Smith, *Conic Sections*, §121, page 123.

If z describe the circle $KF'G'$, the point w will trace out same ellipse as when the point z described the circle FGH. Since these circles are inverse with respect to the unit circle, the image of which is the segment from $(+1, 0)$ to $(-1, 0)$, we see that the cut for the Riemann surface must be along this segment. The points $(\pm 1, 0)$ are the foci of the ellipses, since $[\frac{1}{2}(\frac{1}{r}+r)]^2 - [\frac{1}{2}(\frac{1}{r}-r)]^2 = 1$. Any point P on the unit circle, considered as a limiting case of both sets of circles, has accordingly two images placed exactly opposite each other with respect to the segment from $(+1, 0)$ to $(-1, 0)$, a direct consequence of the sense of description detailed above. Hence the portion of the upper sheet (upon which we shall map the region of the z-plane exterior to the unit circle) which is below the horizontal axis is joined to the portion of the lower sheet (upon which we shall map the region of the z-plane interior to the unit circle) above the horizontal axis. The junction is made along the cut from $(+1, 0)$ to $(-1, 0)$. The other portions of the two sheets, upper and lower, are also joined. The whole of each ellipse lies entirely in one or the other of the two sheets, since the points $(\pm 1, 0)$ are the foci of every ellipse.

Consider next the map of a half-ray ${}_0P_\infty$ through the origin (Fig. 3). As the point z moves along the half ray $\theta = \alpha$ from 0 to ∞, the point $1/z$ moves along the half-ray $\varphi = -\alpha$ from ∞ to 0. As $|z|$ ranges from 0 to 1, $\frac{1}{|z|}$ ranges from ∞ to 1; and as $|z|$ ranges from 1 to ∞, $\frac{1}{|z|}$ ranges from 1 to 0. Hence the points z and $1/z$ describe along the two half-rays two projective ranges of points.

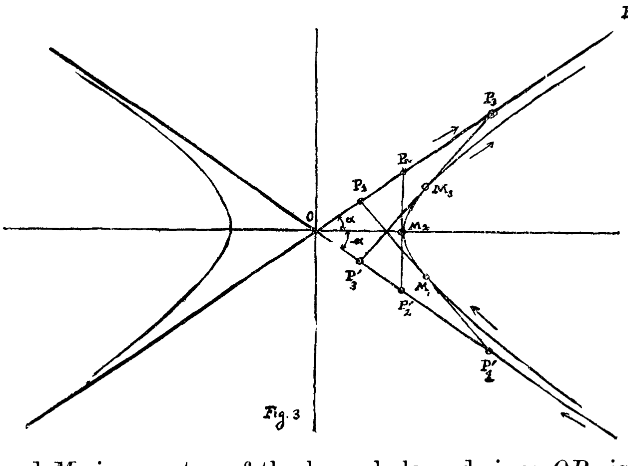


Fig. 3

Since $w = \frac{1}{2}(z + \frac{1}{z})$, the mid-point of any segment PP_1' describes the locus of w. Now the lines PP_1', P_2P_2', ... envelope a conic section which has two points at ∞, one on each half-ray and hence is a hyperbola, to which the half-rays are the asymptotes. When OP_2 is equal to 1, OP_2' is also equal to 1 and M_2 is a vertex of the hyperbola and since OP_2 is distance from the center to the focus, we find that the foci of the hyperbola are $(\pm 1, 0)$. That the system of hyperbolae should be confocal and therefore orthogonal to the former system of ellipses was to be expected, since they were the images of orthogenal loci. The correspondence and sense of description is patent from the drawings, both by means of the lettering and the arrows. For example, let the point z describe

the line EOC beginning at D_∞ in the lower left hand corner. As z moves from D_∞ through E to F, w moves in the upper sheet from D_∞'' through E'' to F''; as

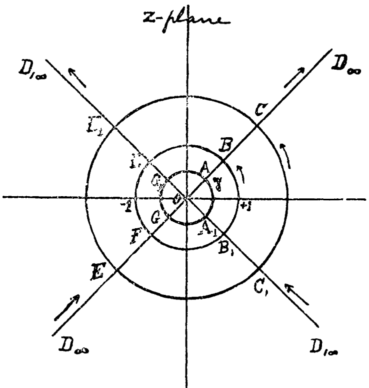


Fig. 4.

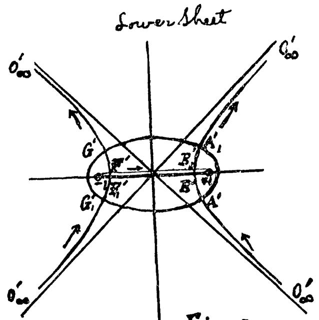


Fig. 5.

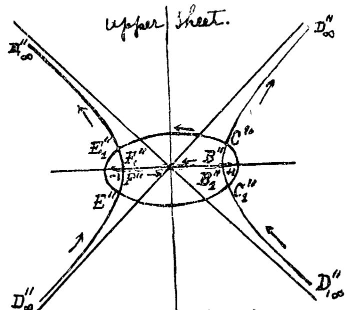


Fig. 6.

z moves from F through G to O, w moves in the lower sheet from F' through G' to O_∞'; as z moves from O through A to B, w moves in the lower sheet from O_∞' lower right hand) through A' to B; and lastly, as z moves from B through C to D_∞, w moves in the upper sheet from B'' through C'' to D_∞''.

THE UNIVERSITY OF CHICAGO, *March, 1903.*

THE GENERAL EUCLIDEAN CONSTRUCTION.

By CHARLES H. SISAM, A. B., Graduate Scholar, Cornell University.

Required to construct, with straight edge and compasses, a length y, given the relation $y=\varphi(a, b, \ldots., l)$; where $a, b, \ldots., l$ represent known lengths and φ is a function involving only rational expressions and square roots.

In order that the construction shall be possible, φ must be homogeneous of first degree. For, let $y_1, a_1, b_1, \ldots., l_1$ be the numerical values $y, a, b, \ldots., l$ for some unit of length z. Then for any other unit of length $(1/k)z$, we must have $ky_1\equiv\varphi(ka_1, kb_1, \ldots., kl_1)$ for all values of k.

Consider one of the radicals of φ such that the expression under it does not contain any radicals. This expression is composed of a sum of terms of the form $\dfrac{P}{Q}a^\alpha b^\beta \ldots. l^\lambda$; where $P, Q, \alpha, \beta, \ldots., \lambda$ are integers; and, for all the terms of the expression $\alpha+\beta+\ldots.+\lambda=c$, a constant, which can—by removing from under the radical sign, if necessary, an even power of any of the quantities $a, b, \ldots., l$—be made equal to 2.

On any line, lay off a convenient length $ON\equiv x$. At N erect a perpendicular and on the perpendicular lay off $NA=a$, $NB=b$, etc. Draw $OA, OB, \ldots, OL$.

Let a be positive. On ON lay off $OA_1=a$. At A_1 erect a perpendicular cutting OA in A_1'. By similar triangles, $A_1A_1'=a^2/x$. On ON lay off $OA_2=A_1A_1'$, and erect a perpendicular cutting OA in A_2'. Then $A_2A_2'=a^3/x^2$; and so on, till we obtain $A_{\alpha-1}A'_{\alpha-1}=a^\alpha/x^{\alpha-1}$. If β is positive, lay of, on ON, $OB_1=A_{\alpha-1}A_{\alpha-1}'$ (or equal to NA if $\alpha=1$). Erect a perpendicular cutting OB in B_1'; and so on, till we obtain $B_\beta B_\beta'=\dfrac{a^\alpha b^\beta}{x^{\alpha+\beta-1}}$. If γ, for example, is negative, draw through B_β' a parallel to ON cutting OC in C_1'. From C_1' drop a perpendicular on ON cutting ON in C_1. Then $OC_1=\dfrac{a^\alpha b^\beta c^{-1}}{x^{\alpha+\beta-2}}$. On NC lay off $NC_2''=OC_1$. Erect a perpendicular cutting OC_2 in C_2'. Drop a perpendicular C_2C_2' on ON. Then $OC_2=\dfrac{a^\alpha b^\beta c^{-2}}{x^{\alpha+\beta-3}}$. Continuing in this way, we obtain OL_λ or $L_\lambda L_\lambda'=\dfrac{a^\alpha b^\beta \ldots . l^\lambda}{x^{\alpha+\beta \ldots . \lambda-1}}=\dfrac{a^\alpha b^\beta \ldots . l^\lambda}{x}$. By laying off OL_λ in succession P times and applying the known construction for the Q section of a line we obtain $SR=\dfrac{P}{Q}\dfrac{a^\alpha b^\beta \ldots . l^\lambda}{x}$.

Making a similar construction for each of the terms under the radical considered and taking the algebraic sum of the resulting lines we obtain $TU=\dfrac{\Sigma\,(P/Q)a^\alpha b^\beta \ldots . l^\lambda}{x}$.

From N lay off on ON, $NV=TU$,—away from O if TU is positive. On OV as a diameter construct a circle cutting NA in M (and M'). Then $NM=\sqrt{\Sigma\,\dfrac{P}{Q}a^\alpha b^\beta \ldots . l^\lambda}=m$ (say).

[We may consider $NM'=-NM$. In this way the conjugate values of y may be constructed].

Hence $y=\varphi_1(a, b, \ldots ., l, m)$, where φ_1 is homogeneous and contains one less radical than φ. Continuing in this way, we obtain finally $y=\psi(a, \ldots ., l, m, \ldots ., r)$, where ψ is a sum of terms of the form $(P'/Q')a^{\alpha'}b^{\beta'} \ldots . r^{\rho'}$ and $\alpha'+\beta' \ldots . +\rho'=1$. Constructing the sum of these terms after the manner above indicat. ed, we obtain the length y required.

A GENERAL THEORY OF PROJECTILES.

By M. E. GRABER, Heidelberg University. Tiffin. Ohio.

There seems to be a lack of uniformity in defining the term *projectile.* Some definitions are too exclusive or special, while others are too inclusive. From a consideration of the conditions involved the following definition seems to answer the purpose well:

A projectile is any body projected from a given point in a given direction subject

to the force of gravity and whatever other forces are inherent in the medium through which the body is projected.

This definition, not being limited to one medium, enables us to apply the term "projectile" not only to particles impelled through the atmosphere but also to particles projected through any medium. This would of course include submarine torpedoes in the category of projectiles.

The method of treatment of the theory of projectiles has generally been first the discussion of motion *in vacuo* and then the discussion of motion in a resisting medium such as the atmosphere. Why not first deduce and discuss formulae of general application to the motion of particles and then consider motion *in vacuo* as a special case with particular values assigned to the different coefficients and factors?

The first consideration in the discussion of the general case is the resistance of the air to the motion of a particle through the atmosphere.

It has been proven by experiment that the resistance of the air encountered by a projectile is proportional to the exposed area. This area is proportional to the square of the diameter of the projectile. Calling this resistance R, we have $R=d^2\phi(v)$, and the corresponding retardation $r=\frac{g}{w}R=g\frac{d^2}{w}\phi(v)$. Putting D (ballistic coefficient) for $\frac{w}{d^2}$, we have $r=\frac{g}{D}\phi(v)$. Assuming that $\phi(v)=B_1 v^n$, in which B_1 and n are constants, to be determined by experiment, we get

$$R=\frac{B_1 d^2 v^n}{g} \text{ and } r=\frac{B_1 v^n}{D}\ldots.(1).$$

Assuming that the axis of the (oblong) projectile lies constantly in the tangent to the trajectory and that the air is calm and of uniform density, the retardation along the tangent at any point of the trajectory due to air resistance is $\frac{B_1 v^n}{D}$; the retardation due to grauity is $g\sin\theta$. Consequently $-\frac{dv}{dt}=\frac{B_1 v^n}{D}+g\sin\theta$. The velocities parallel to the X-axis, which is horizontal forwards, and the Y-axis, which is vertical upwards, are $v\cos\theta$ and $v\sin\theta$. The corresponding retardations are $\frac{1}{D}B_1 v^n\cos\theta$ and $\frac{1}{D}B_1 v^n\sin\theta+g$. Therefore

$$\frac{d(v\cos\theta)}{dt}=-\frac{B_1 v^n}{D}\cos\theta\ldots.(2),$$

$$\frac{d(v\sin\theta)}{dt}=-\left(\frac{B_1 v^n\sin\theta}{D}+g\right)\ldots.(3).$$

Performing the indicated differentiations and comparing the resulting equations we get

$$\frac{vd\theta}{dt}=-g\cos\theta....(4),$$

an expression for the resultant of the forces normal to the direction of resistance. Again, represent the horizontal velocity by v_1; then of course $v_1=v\cos\theta$. Substituting this in (2) and (4), we get

$$\frac{dv_1}{dt}=-\frac{B_1 v_1{}^n}{D\cos^{n-1}\theta}....(5).$$

$$\frac{v_1 d\theta}{dt}=-g\cos^2\theta....(6).$$

The relations between elements of time and elements of the trajectory at any point are given by the following: $dx=v_1 dt$, $dy=v_1\tan\theta dt$, $ds=v_1\sec\theta dt$, where s is the length of the trajectory from the origin. In the above three equations substituting for dt its value $-\frac{v_1}{g}\sec^2\theta d\theta$, we get

$$dx=-\frac{v_1^2}{g}\sec^2\theta d\theta....(7),$$

$$dy=-\frac{v^2}{g}\tan\theta\sec^2\theta d\theta....(8),$$

$$ds=-\frac{v_1^2}{g}\sec^3\theta d\theta....(9).$$

We have now discussed the general theory sufficiently to enable us to derive all the equations of the trajectory *in vacuo*. *In vacuo* the resistance being 0, equation (5) reduces to $dv_1=0$. Consequently v_1 (horizontal velocity) is constant and equal to initial velocity V. Therefore $v\cos\theta=V\cos\phi$. Keeping in mind that ϕ is the initial value of θ, and θ the angle which the tangent of the trajectory at the point x, y makes with the axis of abscissae. Integrating $dt=-\frac{v_1}{g}\sec^2\theta d\theta$ between the limits ϕ and θ, we get

$$t=\frac{V_1}{g}(\tan\phi-\tan\theta)....(10),$$

if $v_1=V_1$. Likewise, by integrating (7), (8), and (9) under the same conditions, we get

$$x=\frac{V_1^2}{g}(\tan\phi-\tan\theta)....(11),$$

$$y=\frac{V_1^2}{2g}(\tan^2\phi-\tan^2\theta)....(12),$$

$$S=\frac{V_1^2}{g}[(\phi)-(\theta)],$$

where $\theta=\int\frac{d\theta}{\cos^3\theta}$ and $\phi=\int\frac{d\phi}{\cos^3\phi}$, both integrals having the same lower limits.

To derive the equation of the trajectory *in vacuo*, eliminate $\tan\theta$ from (11) and (12) by division and addition and we have

$$y=x\tan\phi-\frac{\frac{1}{2}gx^2}{V_1^2},$$

the equation of a parabola whose axis is vertical.

To determine the range, we merely substitute X for x and $-\phi$ for θ in (11); whence $X=\frac{2V_1^2\tan\phi}{g}$. Then since $V\cos\phi=V_1$, $X=\frac{V^2\sin2\phi}{g}$.

To determine the time of flight, set $\theta=-\phi$ in (10). We have

$$T=\frac{2V_1\tan\phi}{g}=\frac{2V\sin\phi}{g}.$$

To determine velocity, since $V\cos\phi=v\cos\theta=V_1$, we have from (12),

$$v^2\sin^2\theta=V^2\sin^2\phi-2gy.$$

Adding $v^2\cos^2\theta$ to the first member and $V^2\cos^2\phi$ (its equal) to the second member, we get $v^2=V^2-2gy$. In a similar manner, we can determine all the properties of a trajectory *in vacuo*.

A PEDAGOGICAL QUESTION IN SPHERICAL TRIGONOMETRY.

By G. W. GREENWOOD. Professor of Mathematics and Astronomy in McKendree College.

The numerous and seemingly disconnected formulae of right and quadrantal spherical triangles are often found, both by student and instructors, difficult to remember and thus detract from the study of spherical trigonometry.

I have found the application of the general formulae for oblique triangles to these triangles very easy and useful, the necessary sets of formulae being

$$\cos a=\cos b\cos c+\sin b\sin c\cos A\ldots.(1),$$

$$-\cos A=\cos B\cos C-\sin B\sin C\cos a\ldots.(2),$$

$$\cot a\sin b=\cos b\cos C+\sin C\cot A\ldots.(3),$$

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}\ldots.(4).$$

The second set can, of course, be written down from the first set by writing $\pi-A$ for a, etc., so that we need remember only the sets (1), (3), and (4). To determine what formula to use in any case, we notice that set (1) connects *three sides and an angle;* set (2), *three angles and a side;* set (3), *two angles and two sides, one of each being included;* set (4), *two angles and two opposite sides.*

We, therefore, choose at once the formula connecting the three given and the one required quantity. In the case of right, or of quadrantal, triangles, one term will always vanish, thus giving the required formula in logarithmic form.

These formulae must sooner or later be learned by the student of spherical trigonometry, and to apply the correct one in any case is easy.

But as an aid in remembering them, we may note that (1) and (3) begin with a *side* and end with the *opposite angle*, the converse being the case in (2). Also in (3) the sides and angles occur in the easily remembered form

$$a,\ b,\ b,\ C,\ C,\ A;$$

and the respective functions occur in the symmetric form

$$\cot,\ \sin,\ \cos,\ \cos,\ \sin,\ \cot.$$

This form of writing (3) I received from the astronomical lectures of Prof. H. H. Turner, of Oxford, and I have found it a very useful way of remembering it.

I think the deduction of the right and quadrantal triangle formulae from these general formulae is economical, and have found it easier to apply them to right triangles than to use the regular formulae by the means of Napier's rules.

They also apply equally well to the quadrantal triangles which are of such frequent occurence in the usual application of spherical trigonometry to elementary astronomy.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

166. Proposed by **F. P. MATZ**, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I sell one of my farms for $\$A,=\$4500$, and the other for $\$B,=\$1800$, I will gain $p\%$, $=5\%$, on cost of both; but if I sell the dearer farm for $\$C,=\$4000$, and the other at cost, I will lose $p\%,=5\%$. Find the cost of each farm.

Solution by G. B. M. ZERR, A. M., Ph.D., The Temple College, Philadelphia, Pa., and J. E. SANDERS, Hackney, Ohio.

$$\frac{\$A+\$B}{1+p}=\frac{\$6300}{1.05}=\$6000=\text{cost of both farms.}$$

$$\$A+\$B-\frac{\$A+\$B}{1+p}=\frac{\$p(A+B)}{1+p}=\frac{\$6300\times.05}{1.05}=\$300=\text{gain.}$$

$$C+\frac{p(A+B)}{1+p}=\frac{C+p(A+B+C)}{1+p}=\$4000+\$300=\$4300=\text{cost of dearer farm.}$$

$$\frac{A+B}{1+p}-\frac{C+p(A+B+C)}{1+p}=\frac{(A+B)(1-p)}{1+p}-C=\$6000-\$4300=\$1700$$

$$=\text{cost of cheaper farm.}$$

Also solved in a similar manner and with same result by *G. W. GREENWOOD.*

ALGEBRA.

171. Proposed by IDA M. SCHOTTENFELTZ, A. M., New York, N. Y.

$ay^2+a=bxy+cx$, $bx^2+b=axy+cy$. Solve for x and y.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$ay^2+a=bxy+cx....(1)$. $bx^2+b=axy+cy....(2)$.

From (1), $x=a(y^2+1)/(by+c)....(3)$.

(3) in (2) gives $[(a^2+b^2)y^2+2bcy+a^2+c^2](cy-b)=0$.

$$\therefore\ y=b/c,\ y=-\frac{1}{a^2+b^2}\{bc\mp a\sqrt{[-(a^2+b^2+c^2)]}\}.$$

$$x=a/c,\ x=-\frac{1}{a^2+b^2}\{ac\pm b\sqrt{[-(a^2+b^2+c^2)]}\}.$$

Also solved by *MARCUS BAKER.*

GEOMETRY.

193. Proposed by PROFESSOR BEYENS.

Si le rapport du segment d'une base de la sphère à l'hemisphère est m/n, le rapport de l'hauteur du segment à deux bases qui resultera au rayon est égal à $2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$. [Problem 9699, *Educational Times.*]

Solution by J. R. HITT, Goss, Miss.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Penn., and G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let R denote radius of sphere, h the altitude of segment of two bases, $R-h$=altitude of segment of one base. Then, $\pi(R-h)^2[R-\frac{1}{3}(R-h)]/\frac{2}{3}\pi R^3$

$=3(R-h)^2(2R+h)/6R^3=m/n$. Therefore, $(h/R)^3-3(h/R)+2=2m/n$; or, $(h/R)^3-3(h/R)+2(n-m)/n=0$. Applying the proper trigonometric formula, $h/R=2\sqrt{(p/3)}\sin\frac{1}{3}\theta$, where $p=3$, $\theta=\sin^{-1}3q/2p\sqrt{(3/p)}=\sin^{-1}(n-m)/n$, $q=2(n-m)/n$. Hence, $h/R=2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$.

194. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Glass paper weights, having the form of a regular tetrahedron, are to be packed for shipment, each in a paper box. Wanted to know the size and shape of the smallest box for the purpose. How much empty space in each box?

Solution by the PROPOSER.

Shape of box = cube.

Edge of box $=s\sqrt{\frac{1}{2}}$, where s=edge of tetrahedron.

Empty space $=\frac{1}{3}s^3\sqrt{\frac{1}{2}}$.

Occupied space $=\frac{1}{6}s^3\sqrt{\frac{1}{2}}$=volume of tetrahedron.

Total space $=\frac{1}{2}s^3\sqrt{\frac{1}{2}}$=volume of box.

Of the twelve diagonals in the six faces of the box, the six edges of the tetrahedron coincide with one in each face.

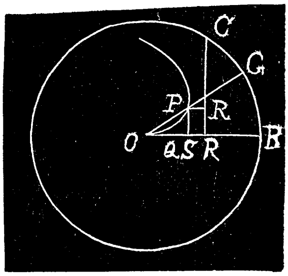
CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly towards the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let the origin be the vertex of the cone, O the center of the base of the cone, σ=the length of the dog's path, s=the length of the projection of the dog's path on the plane (x, y) or the base of the cone, r= radius vector of this projection, a=100 feet=altitude =radius of base, m=6 feet per second, n=5 feet per second, $n/m=u$, $x^2+y^2=z^2$ is the equation of the cone. Then $u\sigma=a\theta$, where $\theta=\angle COB$, subtended by the fox's path at the center O.



$$d\sigma=(a/u)d\theta=\sqrt{(dx^2+dy^2+dz^2)}=\sqrt{(ds^2+dz^2)}$$

$$=[r^2+(dr/d\theta)^2+(dz/d\theta)^2]^{\frac{1}{2}}\,d\theta.$$

But $r^2=x^2+y^2=z^2$. $\therefore\ dz=dr$.

$\therefore\ d\sigma/d\theta=a/u=[r^2+2(dr/d\theta)^2]^{\frac{1}{2}}$, or $d\theta/dr=\left(\dfrac{2u^2}{a^2-r^2u^2}\right)^{\frac{1}{2}}=A$.

Let P be the projection of the dog's path on the plane $(x,\ y)$ at any time; C, the position of the fox at the same time; QC, the tangent at P; $OS=x$; $PS=y$; $OR=t$; $PC=l$; $OP=r$; $\angle POS=\varphi$; $\angle CQS=\psi$. Then $u\sigma=a\theta=a\cos^{-1}(t/a)=a\cos^{-1}[(x+PR')/a]$. But $PR'\,:\,CR'=QS\,:\,PS$. $\therefore\ PR'=l\cos\psi=ldx/ds$.

$\therefore\ u\sigma=a\theta=a\cos^{-1}[(x+ldx/ds)/a]$ or $a\cos(u\sigma/a)=x+ldx/ds$.

Also $a\cos\theta=x+ldx/ds$. But $a^2=r^2+l^2-2lr\cos OPC=r^2+l^2+2lr\cos(\varphi-\psi)$, $\cos(\varphi-\psi)=dr/ds$. $\therefore\ a^2=r^2+l^2+2lrdr/ds$.

$\therefore\ l=\sqrt{[a^2-r^2+r^2(dr/ds)^2]}-rdr/ds$.

But $x=r\cos\varphi$, $y=r\sin\varphi$, $ds/dr=\sqrt{[1+r^2(d\varphi/dr)^2]}$.

$$\frac{dy}{dx}=\frac{\sin\varphi dr/d\varphi+r\cos\varphi}{\cos\varphi dr/d\varphi-r\sin\varphi},\quad \frac{d\varphi}{dr}=\frac{xdy/dx-y}{\sqrt{(x^2+y^2)}(x+ydy/dx)}$$

$$\text{and}\ \frac{ds}{dr}=\frac{\sqrt{\{(x^2+y^2)[1+(dy/dx)^2]\}}}{x+ydy/dx}.$$

$$\therefore\ l=\frac{\sqrt{\{a^2[1+(dy/dx)^2-(y-xdy/dx)^2]\}}-(x+ydy/dx)}{\sqrt{[1+(dy/dx)^2]}}.$$

$$\therefore\ a\cos(u\sigma/a)=a\cos\theta=x+\frac{\sqrt{\{a^2[1+(dy/dx)^2-(y-xdx/dx)^2]\}}-(x+ydy/dx)}{1+(dy/dx)^2}.$$

We also have $l^2=a^2+r^2-2ar\cos(\theta-\varphi)$. This establishes a relation between θ and φ; but it does not simplify the operation any. The differential equation above is too complicated for solution. If we assume that $\theta=\varphi$, and therefore, that the dog is always on a straight line between the fox and the vertex of the hill, the distance he runs can be found; for $d\varphi/dr=d\theta/dr=\dfrac{xdy/dx-y}{r(x+ydy/dx)}=A$.

$l=a-r$, $x=r\cos\theta=r\cos(u\sigma/a)$, $y=r\sin\theta=r\sin(u\sigma/a)$. Let $u\sigma/a=\beta$.

$$\text{Then}\ \frac{dy}{dx}=\frac{Arx+y}{x-Ary}=\frac{Ar\cos\beta+\sin\beta}{\cos\beta-Ar\sin\beta}.\quad dx/ds=\frac{\cos\beta-Ar\sin\beta}{\sqrt{(1+A^2r^2)}}.$$

Then $a\cos(u\sigma/a)=x+ldx/ds$ becomes

$$a\cos\beta=r\cos\beta+\frac{(a-r)(\cos\beta-Ar\sin\beta)}{\sqrt{(1+A^2r^2)}}.\quad \therefore\ \frac{\cos\beta}{\cos\beta-Ar\sin\beta}=\frac{1}{\sqrt{(1+A^2r^2)}}.$$

$$\therefore\ \tan\beta=\frac{1-\sqrt{(1+A^2r^2)}}{Ar}\ \text{or}\ \beta=\tan^{-1}\left(\frac{1-\sqrt{(1+A^2r^2)}}{Ar}\right).\quad \text{Now}\ \beta=u\sigma/a,$$

$$A=\frac{u\sqrt{2}}{\sqrt{(a^2-r^2u^2)}}. \quad \therefore\ \sigma=\left[\frac{a}{u}\tan^{-1}\left(\frac{\sqrt{(a^2-r^2u^2)}-\sqrt{(a^2+r^2u^2)}}{ru\sqrt{2}}\right)\right]_{r=0}^{r=a}.$$

$$\therefore\ \sigma=\frac{a}{u}\tan^{-1}\left(\frac{\sqrt{(1-u^2)}-\sqrt{(1+u^2)}}{u\sqrt{2}}\right)=120\tan^{-1}\left(\frac{\sqrt{11}-\sqrt{61}}{5\sqrt{2}}\right)$$

$$=314\tfrac{1}{3} \text{ feet, nearly.}$$

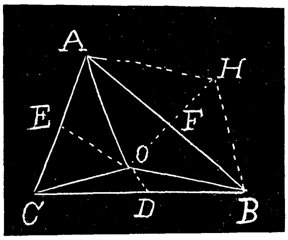
Prof. G. W. Greenwood solves the problem, interpreting it according to Prof. Zerr's assumption, but he gets a slightly different result. The Proposer furnished a partial solution for *The Educational Times*, London, obtaining the same differential equation as that obtained by Prof. Zerr, but by a different method.

MECHANICS.

150. Proposed by G. B. M. ZERR, A. M., Ph. D, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

O is a point in a plane of a triangle, ABC, and D, E, F are the mid-points of the sides. Show, geometrically, that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

Solution by G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.



Complete the parallelogram $AOBH$, of which OE is one-half the diagonal.

Since the forces OH, OB, are equivalent to OH, $\frac{1}{2}OA$ and $\frac{1}{2}OB$ will be equivalent to OF.

Similarly, $\frac{1}{2}OB$ and $\frac{1}{2}OC$ will be equivalent to OD, and $\frac{1}{2}OC$ and $\frac{1}{2}OA$ will be equivalent to OE.

Hence the system OA, OB, OC is equivalent to the system OD, OE, OF.

151. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

An elastic ball is projected along a horizontal tube, striking first the bottom, then the top, then the bottom and so on. Find the number of times the ball will strike the top.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let v=velocity just before first impact, r=diameter of tube, $v_1=\sqrt{(2gr)}$, e=coefficient of restitution. Then velocity of leaving bottom$=ev$, velocity of arrival at top$=ev-v_1$, velocity of arrival at bottom$=e^2v-ev_1+v_1$, velocity of second arrival at top$=e^3v-e^2v_1+ev_1-v_1$, velocity of third arrival at top$=e^5v-e^4v_1+e^3v_1-e^2v_1+ev_1-v_1$.

$\therefore$ Velocity of nth arrival at top$=e^{2n-1}v-v_1(e^{2n-2}-e^{2n-3}+e^{2n-4}-e^{2n-5}+\dots+e^4-e^3+e^2-e+1)$, and this velocity$=v_1$.

$\therefore\ e^{2n-1}v-v_1(e^{2n-2}+e^{2n-4}+\dots+e^4+e^2)+v_1(e^{2n-3}+e^{2n-5}+\dots+e^3+e)=2v_1$.

$\therefore\ e^{2n-1}v-e^2v_1\left(\dfrac{e^{2n-2}-1}{e^2-1}\right)+ev_1\left(\dfrac{e^{2n-2}-1}{e^2-1}\right)=2v_1.$

$\therefore\ v(e^2-1)e^{2n-1}-ev_1e^{2n-1}+v_1e^{2n-1}=v_1(e^2+e-2).$

$\therefore\ e^{2n-1}=\dfrac{v_1(e+2)}{v(e+1)-v_1}=A$, suppose.

$\therefore\ 2n-1=\log A/\log e.\quad \therefore\ n=\frac{1}{2}\log(Ae)/\log e$=the number required.

DIOPHANTINE ANALYSIS.

107. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three positive integral numbers such that the sum of all three of them, and the sum of every two of them shall be a square number.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x, y, and z=the numbers.

Then $x+y=\square=h^2$, $x+z=\square=k^2$, $y+z=\square=l^2$.

$\therefore\ x+y+z=\frac{1}{2}(h^2+k^2+l^2)=\square=s^2\ldots.(1)$.

$\therefore\ x=s^2-l^2$, $y=s^2-k^2$, and $z=s^2-h^2$.

Put $l=s-m$, $k=s-n$, and $h=s-r$. Substituting these values in (1), we obtain $3s^2-2s(m+n+r)+m^2+n^2+r^2=2s^2$.

Solving for s, we find $s=m+n+r\pm\sqrt{[2(mn+mr+nr)]}$.

Take $mn+mr+nr=mn+r(m+n)=2b^2$.

$\therefore\ s=m+n+x\pm2b$,

$x=2ms-m^2=m(2s-m)\ldots.(2)$,

$y=2ns-n^2=n(2s-n)\ldots.(3)$,

$z=2rs-r^2=r(2s-r)\ldots.(4)$.

Put $n=m+a$. Then $mn+r(m+n)=m(m+a)+r(2m+a)=2b^2$.

$\therefore\ r=\dfrac{2b^2-m(m+a)}{2m+a}$, and $s=\dfrac{(m+a)(3m+a)-am+2b[b\pm(2m+a)]}{2m+a}$.

Substituting in (2), (3), and (4), and multiplying by $(2m+a)^2$, we obtain the following general values:

$$x=m(2m+a)\{2(m+a)(2m+a)-am+4b[b\pm(2m+a)]\},$$
$$y=(m+a)(2m+a)\{(2m+a)^2-am+4b[b\pm(2m+a)]\},$$
$$z=[2b^2-m(m+a)]\{2[b\pm(2m+a)]^2-m(m+a)\},$$
$$x+y=\{(2m+a)[2b\pm(2m+a)]\}^2,$$
$$x+z=\{m^2+2b[b\pm(2m+a)]\}^2,$$
$$y+z=\{(m+a)^2+2b[b\pm(2m+a)]\}^2,$$
$$x+y+z=\{(m+a)(3m+a)-am+2b[b\pm(2m+a)]\}^2.$$

For *positive* values, we have the general condition, $2b^2>m(m+a)$; also, when $b-(2m+a)$ is used, the condition, $b>2m+a$.

When x, y, and z have a common divisor, lowest values are obtained by dividing by the highest common *square* factor.

Multiple values may be obtained by multiplying any set of values of x, y, and z by a *square* number.

m, a, and b may be any integers, subject to the conditions for positive values. $a=0$, when $m=n$.

The least values are obtained by taking $m=a=1$, and $b=2$, and dividing by 3^2, the highest common square factor.

Whence $x=17$, $y=32$, $z=32$.

These values may also be found by taking $a=0$, and $m=b=1$. Then $x=32$, $y=32$, $z=17$.

The least *different* values are obtained by taking $m=a=1$, and $b=4$, using $b+2m+a$, and dividing by 3^2.

Whence $x=41$, $y=80$, $z=320$.

By using $b-(2m+a)$, in the last case, we find $x=9$, $y=16$, $z=0$.

Excellent solutions of this problem were received from *PROFESSORS ZERR*, *CROSS*, and *WALKER*, and the late *JOSIAH H. DRUMMOND*. Mr. Cross sent in two solutions, one of which was a solution of the generalized problem. If space permits, his solution of the generalized problem will be published in the next issue of the MONTHLY.

Professor Walker should have been credited with a solution of problem 106.

No solutions of problems 105 and 108 have yet been received.

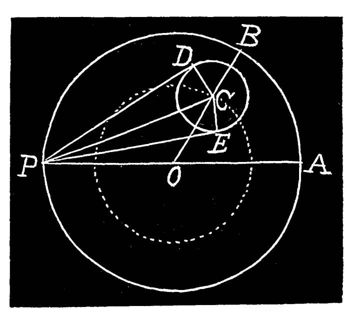
AVERAGE AND PROBABILITY.

90. Proposed by WALTER H. DRANE. Graduate Student, Harvard University.

During a rain-storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond?

III. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the following solution, we assume that the path of the man is along a random straight line drawn from a random point in the circumference of the field. We also assume that the number of favorable paths is to the total number of paths as the arc of a circle (radius, the line drawn from the random point on the circumference of the field to the center of the pond) intercepted by the pond, is to the semi-circumference of the same circle, and that all directions of the path are equally probable; that all values of the radius of the pond less than the radius of the field are also equally probable; that all points on the circumference of the field are equally likely to become the point of starting across the field; and that all points of the field are equally likely to become the center of the pond.



Let O be the center of the field, radius $AO=R$; C, the center of the pond; and P, the point where the man enters the field.

Let $x=OC$, the distance from the center of the field to the center of the pond; $z=CD=CE$, the radius of the pond; $\theta=\angle AOB$; and $\phi=\angle CPE=\angle CPD=\sin^{-1}\left(\dfrac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)$

Then, (1) the chance that the center of the pond lies on the area comprised between two

concentric circles whose common center is O and whose radii are x and $x+dx$ is $C_1=\frac{2\pi x dx}{\pi R^2}=\frac{2xdx}{R^2}$; (2) the chance that the radius of the pond lies between z and $z+dz$ is $C_2=\frac{dz}{R-x}$; (3) the chance that the line of centers, OC, makes an angle with the diameter, AP, between θ and $\theta+d\theta$ is $C_3=d\theta/\pi$; (4) the chance that the path of the man lies between ϕ and $\phi+d\phi$ is $C_4=d\phi/\pi$.

The chance of the concurrence of all of these events is $C_0=C_1C_2C_3C_4$, and the chance of the concurrence of all these events for all values of the variables is

$$C=\int C_1\int C_2\int C_3\int C_4$$

The limits of θ are 0 and π and doubled; the limits of x are 0 and R; the limits of z are 0 and $R-x$; and the limits of ϕ are 0 and ϕ.

$$\therefore\ C=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{dz}{R-x}\int_0^\phi d\phi=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R xdx\int_0^{R-x}\frac{dz}{R-x}\phi$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R xdx\int_0^{R-x}\frac{1}{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)dz$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R\frac{xdx}{R-x}\left[z\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)+\sqrt{R^2+x^2+2Rx\cos\theta}\right]_0^{R-x}$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R[x\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)+2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta$$

$$-\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}]\,dx=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\left[\tfrac{1}{2}x^2\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)\right]_0^R d\theta$$

$$+\tfrac{1}{2}\int_0^R dx\int_0^\pi\frac{x(R+x)\sqrt{(Rx)}\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}d\theta+2\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}\int_0^\pi\cos\tfrac{1}{2}\theta d\theta$$

$$-\int_0^\pi d\theta\int_0^R\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx=\frac{2}{\pi^2R^2}\left\{\int_0^R x\log\left(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right)dx\right.$$

$$\left.+4\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}-\int_0^\pi d\theta\int_0^R\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx\right\}$$

$$=\frac{2}{\pi^2R^2}\left\{\tfrac{1}{2}x^2\log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right]_{x=R}-\tfrac{1}{2}\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}+4\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x} \sqrt{(R^2+x^2+2Rx\cos\theta)}dx \Big\} = \frac{2}{\pi^2 R^2}\Big\{ \Big[\tfrac{1}{2}x^2 \log \frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$+\frac{z}{2}\sqrt{R}\Big[-\tfrac{2}{3}x^{\frac{3}{2}} - 2Rx^{\frac{1}{2}} + R^{\frac{3}{2}}\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]\Big]_0^R$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x} \sqrt{(R^2+x^2+2Rx\cos\theta)}\, dx \Big\}$$

$$= \frac{2}{\pi^2 R^2}\Big\{ -\tfrac{2}{3}{}^8 R^2 + \Big[\tfrac{1}{2}(x^2+7R^2)\log \frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x} \sqrt{(R^2+x^2+2Rx\cos\theta)}\, dx \Big\}$$

$$= \frac{2}{\pi^2 R^2}\Big\{ -\tfrac{2}{3}{}^8 R^2 + \Big[\tfrac{1}{4}(x^2+7R^2)\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]_{x=R} \int_0^\pi \cos(\tfrac{1}{2}\theta)d\theta$$

$$+\int_0^\pi d\theta \Big[\tfrac{1}{2}(x+R\cos\theta)\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$+\tfrac{1}{2}R^2\sin^2\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+x+R\cos\theta]+R\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$-2R^2\cos^2\tfrac{1}{2}\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+R-x-2R\cos^2\tfrac{1}{2}\theta]-2R^2\cos\tfrac{1}{2}\theta\times$$

$$\log\Big(\frac{\sqrt{(R^2+x^2+2Rx\cos\theta)}+2R\cos\frac{1}{2}\theta)}{R-x}-\cos\tfrac{1}{2}\theta\Big)\Big]_0^R$$

$$= \frac{2}{\pi^2 R^2}\Big[-\tfrac{2}{3}{}^8 R^2 + R^2\int_0^\pi (2\cos^2\tfrac{1}{2}\theta - \tfrac{1}{2}\cos\theta + 2\cos\tfrac{1}{2}\theta - 1)d\theta$$

$$+R^2\int_0^\pi (\tfrac{1}{2}\sin^2\theta + 2\cos^2\tfrac{1}{2}\theta + 2\cos\tfrac{1}{2}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta\Big]$$

$$= \frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_0^\pi$$

$$-\int_0^\pi (-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\frac{\sin(\frac{1}{2}\theta)d\theta}{2\cos\frac{1}{2}\theta(1+\cos\frac{1}{2}\theta)}\Big\},$$

$$= \frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta=\pi}+\tfrac{1}{4}\int_0^\pi (1-\cos\tfrac{1}{2}\theta)\cos\theta d\theta$$

$$-\int_0^\pi (1-\cos\tfrac{1}{2}\theta)d\theta - 2\int_0^\pi \Big(\frac{1-\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta - \tfrac{5}{4}\int_0^\pi \frac{\theta(1-\cos\frac{1}{2}\theta)d\theta}{\sin\theta}\Big\},$$

$$=\frac{2}{\pi^2}\Big[-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta\pm\pi}-\tfrac{1}{6}-(\pi-2)-\Big[4\log\Big(\frac{1+\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)$$

$$-2\theta\Big]_{\theta\pm\pi}-\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{2}\theta\Big]_0^\pi-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{2}\theta)d\theta^*+\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{4}\theta\Big]_0^\pi$$

$$-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{4}\theta)d\theta\Big\}=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log2-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{4}\theta)d\theta\Big)=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log2$$

$$-5\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta\Big)=\frac{2}{\pi^2}\Big(40\sum_{n=1}^{n=\infty}\frac{2n-1}{(4n-3)^2(4n-1)^2}-\tfrac{5}{6}-4\log2\Big)$$

$$=\frac{2}{\pi^2}(40\times.114488335\dagger-\tfrac{5}{6}-4\log2)=\frac{2}{\pi^2}(4.5795334-\tfrac{5}{6}-4\log2).$$

NOTE.—The above solution of the "pond problem" was attempted at the same time that the published solution of this problem was prepared, but owing to an error in finding an approximate value of

$$\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta$$

the result obtained, appeared to be negative. The solution of problem 156, Calculus Department, involved the evaluation of this same definite integral and in finding the approximate value of this integral, again the result was put in the former answer of the "pond problem" and thus led to a consistent answer.

At the time the former solutions of this problem were published it was my intention to publish this solution, for the reason that about thirty years ago, a controversy arose between Mr. Henry Heaton, of Atlantic, Iowa, and Dr. Joel E. Hendricks, Editor of the *Analyst*, concerning the method of solution of problem 135, of the *Analyst*, this problem involving the same point of difference of solution as is involved in the above solution and my former solution.

In this controversy, Mr. Heaton championed the method followed out in the above solution, and Dr. Hendricks defended the method of the two former published solutions. Mr. Heaton tells us that Professor E. B. Seitz was first with Dr. Hendricks, but was finally convinced by the argument of Mr. Heaton conveyed to him through correspondence. Professor Seitz then converted Dr. Hendricks.

The controversy was again renewed when in 1877, Mr. Heaton sent to Dr. Artemas Martin, now of Washington, D. C., a solution of problem 27, *Mathematical Visitor*, a journal edited and published by Dr. Martin himself,—this problem involving the same bone of contention. Mr. Heaton says, "When I sent Mr. Martin my solution, he wrote back that it was wrong, and said that he had a correct solution from another contributor, and that Professor Benjamin Pierce, and Professor Woolhouse, of England, had decided my solution wrong. I immediately wrote Professor Pierce inclosing Mr. Martin's letter and my solution." In reply to this letter, Mr. Heaton received the following letter from Professor Pierce, the original of which Mr. Heaton kindly loaned me:

MY DEAR SIR:

Mr. Martin informed you correctly concerning my decision. But upon reconsideration, I have decided that I must reverse it, and I now regard your solution as correct. I shall write the grounds of my reversal to Mr. Martin.

Yours faithfully,

BENJAMIN PIERCE.

Thus the controversy ended, and many problems in probability have been solved indifferently by one method or the other from that time to the present.

The solution of the "pond problem" again raises the question as to which method is correct. Certainly, both methods can not be correct, since they lead to different results, when the assumptions as to the distribution of the several events are granted in any particular way.

In the first place, the problem is indefinite, since as many different solutions may be obtained as there are ways of interpreting it and assigning the laws of distribution of the several events. When these have once been assumed then all solutions by whatever method should lead to the same result.

Suppose that the distribution of the events are assumed as in the solution above. Then the solution

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*See problem 123, Calculus Department.

†This is the value computed by Professor Zerr, problem 156, Calculus Department.

of the problem is based upon the definition of the probability of an event which is the *total number of favorable cases divided by the total number of cases.*

In this problem, the favorable cases have been assumed to be proportional to the length of the arc $\phi\times PE$ and the total number of cases proportional to the length of the semi-circumference $\pi\times PE$.

Hence, the chance that the man crosses the pond for any particular value of ϕ, x and θ being constant, is

$$C_1=\frac{\phi\times PE}{\pi\times PE}=\phi/\pi=\frac{\sin^{-1}[z\div(R^2+2x^2+2Rx\cos\theta)]}{\pi}=\frac{f(z)}{\pi}.$$

Now, all values of $f(z)$ between the limits $z=0$ and $z=R-x$ must be considered. Let

$$h=\frac{0+R-x}{n}=\frac{R-x}{n}.\quad \text{Then,}\ \sum_{z=0}^{z=R-x}f(z)=f(h)+f(2h)+f(3h)+\ldots.+f(nh).$$

The chance that $f(z)$ has the value of any particular term as $f(kh)$ of this series is $1/n$, or $h/(R-x)$.

Hence, the chance that the man crosses the pond is the product of the chance that $f(kh)$ is the value of $f(z)$ multiplied by $f(kh)/\pi=f(kh)/[\pi(R-x)]$.

Hence, the chance that the man crosses the pond for all values of $f(z)$ between the limits $z=0$ and $z=R-x$, x and θ being constant, is

$$C_2=\frac{f(h)+f(2h)h+f(3h)h+\ldots+f(nh)h}{\pi(R-x)}=\frac{\int_0^{R-x}f(z)dz}{\pi(R-x)}=F(x).$$

$F(x)$ is the chance of the man crossing the pond when the center of the pond is at a certain fixed point. But the center of the pond may be any point in the field. The chance that the center of the pond falls on any point between two concentric circles whose radii are x and $x+dx$ is $\pi xdx/\pi R^2=2xdx/R^2$.

Hence, the chance of the man crossing the pond is $(2xdx/R^2)F(x)$, and for all values of x between $x=0$ to $x=R$ the chance is

$$C_3=\int_0^R\frac{2xdx}{R^2}F(x)=\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{f(z)dz}{\pi(R-x)}=\Psi(\theta).$$

The chance that θ has any value between θ and $\theta+d\theta$ is $d\theta/\pi$.

Hence, the chance that the man crosses the pond for all values of $\psi(\theta)$ between $\theta=0$ to θ is

$$C_4=\int_0^\pi\frac{d\theta}{\pi}\Psi(\theta)=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}F(x)=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{f(z)dz}{\pi(R-x)}.$$

Thus it appears that the first method when made to obey strictly the definition of probability leads to the same form for computation as the second. The first two solutions of this problem as published in the December Number, of Vol. VII, of the MONTHLY, are, therefore, incorrect.

Dr. Martin informs me that Professor Seitz sent him a solution of this problem and obtained the same result as that given in the two solutions previously published. Professor Seitz, in his day, solved a great many very difficult probability problems, the solutions of very few of which have since been found erronous, if the laws of distribution assumed by him be granted.

The above solution of the "pond problem" has the sanction of the ablest living mathematicians in this country.

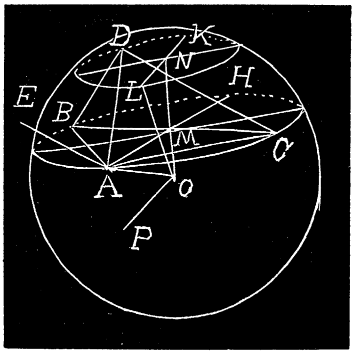
This point, viz., that the total number of events are not to be obtained in such problems by writing for the denominator a certain set of definite integrals, should not be overlooked in solving probability problems of this particular in the future.

130. Proposed by **LON C. WALKER, A. M.**, Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the the points taken three and three, is 1-35 of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r; $ABFE$ a great circle through A, B; ABC a small circle through A, B, C with center S; DGF a small circle through D parallel to ABC with center P; M the middle point of AB.



Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$AM=r\sin\frac{1}{2}\theta=r_1\cos\psi_1$,

$SM=r\cos\frac{1}{2}\theta\cos\phi=r_1\sin\psi_1$,

$OS=r\cos\frac{1}{2}\theta\sin\phi$,

$AC=2r_1\cos(\psi-\psi_1)$,

$PD=\sqrt{(r^2-x^2)}$,

$r_1=r(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta\cos^2\phi)^{\frac{1}{2}}$, area $ABC=2rr_1\sin\frac{1}{2}\theta\sin\psi\cos(\psi-\psi_1)$,

volume of tetrahedron $D-ABC=\frac{1}{3}SP.\text{area}ABC=\frac{2}{3}rr_1(x+r\cos\frac{1}{2}\theta\sin\phi)\sin\frac{1}{2}\theta$ $\times\sin\theta\cos(\psi-\psi_1)$. Hence we have for the required average volume

$$V=\frac{1}{(4\pi r^2)^3}\int_0^\pi\int_0^\pi\Bigg[\int_0^{\frac{1}{2}\pi+\psi_1}\Bigg(\int_{-r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x+r\cos\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi-\psi_1).2\pi rdx$$

$$+\int_{r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x-r\cos\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi-\psi_1).2\pi rdx\Bigg)4rr_1\sin\psi\cos(\psi-\psi_1)d\psi$$

$$+\int_0^{\frac{1}{2}\pi-\psi_1}\Bigg(\int_{-r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x+r\mathrm{cys}\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi+\psi_1).2\pi rdx$$

$$+\int_{r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x-r\cos\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi+\psi_1).2\pi rdx\Bigg)4rr_1\sin\psi\cos(\psi+\psi_1)\Bigg]d\phi$$

$$\times 2\pi r^2\sin\theta d\theta$$

$$=\frac{r}{3}\int_0^\pi\int_0^\pi\Bigg[\int_0^{\frac{1}{2}\pi+\psi_1}r_1\sin^2\psi\cos^2(\psi-\psi_1)d\psi+\int_0^{\frac{1}{2}\pi+\psi_1}r_1^2\sin^2\psi\cos^2(\psi+\psi_1)d\psi\Bigg]$$

$$\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta(1+\cos^2\tfrac{1}{2}\theta\sin^2\phi)d\theta d\phi$$

$$=\frac{r^3}{24}\int_0^\pi\int_0^\pi(\sin^2\tfrac{1}{2}\theta+3\cos^2\tfrac{1}{2}\theta\cos^2\phi)(1+\cos^2\tfrac{1}{2}\theta\sin^2\phi)\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta d\theta d\phi$$

$$=\frac{r^3}{24}\int_0^\pi(\tfrac{32}{35}+\tfrac{12}{35}\cos2\phi-\tfrac{2}{35}\cos4\phi)d\phi=\tfrac{4}{105}\pi r^3=\tfrac{1}{35}\text{ of the volume of given sphere.}$$

MISCELLANEOUS.

130. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

From a cloud of angular elevation $\phi=45°$, a streak of lightning darted to the earth. The temperature of the atmosphere was $t=80°$, and the percentage of humidity $p=90$. After $m=3$ seconds, the report of the stroke at the earth was heard. How far away from the observer did the streak of lightning (1) start, and (2) strike the earth?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let H=tension of aqueous vapor at $t°=80°$.

Let H'=tension of aqueous vapor at dew point.

Then $(H'/H)100=p=90$, $80°F=26°.7C$.

Now $H=26.045$mm. $\therefore$ $H'=\frac{9}{10}$ of 26.045mm.=23.4405mm.

Let us take the barometric pressure at 740mm.=P.

Then 740mm.$-$23.4405mm.=716.5)95mm. will represent the pressure of dry air.

Let W=weight of litre of dry air at $80°F$.

W'=weight of vapor in one litre.

$$W=1.293187\times\frac{716.5595}{760}\times\frac{1}{1+.00367\times26.7}=1.110667 \text{ grams.}$$

$$W'=1.293187\times .6235\times\frac{23.4405}{760}\times\frac{1}{1+.00367\times26.7}=.022653 \text{ grams.}$$

$W+W'==1.13332$ grams=total weight of litre.

$V=\sqrt{(Pk/\rho)}$, where $k=1.41$=ratio of specific heat of air at constant pressure to the specific heat at constant volume.

$P=740$mm.=986634 dynes per square cm., ρ=density=.00113332.

$\therefore$ $V=\sqrt{[(986634\times1.41)/.00113332]}=350.36$ meters per second.

$3\times60\times350.36=63.0648$ kilometers=distance of observer from point on earth where lightning strikes. If we regard the distance as a straight line, then distance to cloud=$63.0648\sqrt{2}=89.18624$ kilometers.

If we regard the earth as a sphere, diameter 6370.946 kilometers, then $1°=111\frac{1}{9}$ kilometers. Therefore $34'\ 3.3''$=63.0648 kilometers.

$$\text{Distance to cloud}=\frac{6370.946\sin(34'\ 3.3'')}{\sin(44°\ 25'\ 56.7'')}=91.4594 \text{ kilometers.}$$

131. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Find a power series for π^{nx} (n=any integer).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$\pi^{nx}=1+nx\log_e\pi+\frac{n^2x^2}{2!}(\log_e\pi)^2+\frac{n^3x^3}{3!}(\log_e\pi)^3+\ldots.$$

Now $\log_e\pi = \log_e[1+(\pi-1)] = (\pi-1) - \frac{1}{2}(\pi-1)^2 + \frac{1}{3}(\pi-1)^3 - \frac{1}{4}(\pi-1)^4 + \ldots = A$, suppose.

$\therefore\ \pi^{nx} = 1 + Anx + (Anx)^2/2! + (Anx)^3/3! + (Anx)^4/4! + \ldots$

NOTE.—Frank Gilman, of Churchville, N. Y., sent in a solution of problem 129, which is based on Chauvenet's method. The method has the advantage of being more convenient in practice and lends itself more easily to logarithmic computation. We have not the space to publish the solution. Mr. Gilman gets as a result 57° 38′ 17″.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A teacher's monthly salary after $m=2$ increases of $p=20$ and $q=10\%$, is $\$M=\$120$. What was the original salary?

ALGEBRA.

177. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $m^{2x}(m^2+1) = (m^{3x}+m^x)m$.

GEOMETRY.

200. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of eight points of contact of the four common tangents of two concentric coaxial ellipses.

CALCULUS.

165. Proposed by CAPT. T. C. DICKSON, Ordnance Department United States Army, Washington.

Solve by integration, the differential equation

$$\frac{d^2\varphi}{dt^2} + \frac{A}{B}\left(\frac{d\varphi}{dt}\right)^2 - \frac{C}{B} = 0, \text{ in which:}$$

$A = 1{,}103{,}430{,}032.196\sin\varphi\cos\varphi - 38{,}579{,}566.1706\sin^2\varphi + 38{,}575{,}641.7961\cos^2\varphi - 310.6332\cos\varphi + 204.6506\sin\varphi + 17.6818M\cos\varphi\sin\varphi + .4082M\sin^2\varphi - .4117M\cos^2\varphi - .3117M\sin\varphi + .0061M\cos\varphi$,

$B = 6382.5395\sin\varphi\cos\varphi + 59{,}363.1172\sin^2\varphi - .0095M\sin^2\varphi - 204.65\cos\varphi - -310.6332\sin\varphi - .8199M\cos\varphi\sin\varphi - .0095M\sin^2\varphi - 17.6904M\cos^2\varphi + .0061M\sin\varphi + .3117M\cos\varphi - 1310.866$,

$C = \sin\varphi(16{,}209.4583 - .0029254M) + \cos\varphi(101{,}111.3767 - .12678\,M) - 1080.2307 + .001874M.$

For practical application, it is desired to find the mass M which is required to rotate certain known parts of a machine through the angle from $\varphi = 13^\circ$ to $\varphi = 88^\circ$ in the time $t = 2''$.

MECHANICS.

155. Proposed by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests. Find the position of equilibrium.

156. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

Three perfectly elastic particles start from the cusp of a smooth cycloid (axis vertical, vertex down) at intervals of t seconds. How long will it be to the nth collision?

DIOPHANTINE ANALYSIS.

115. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three square integral numbers the difference between the sum of every two of them and the third shall be a square number.

AVERAGE AND PROBABILITY.

142. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., Washington, D. C.

Two points are taken at random in the arc of a semi-circle, and a third point anywhere in its base. Find the probability that the triangle formed by joining them is acute.

[Unsolved Problem 9955, *Educational Times*, London.]

MISCELLANEOUS.

137. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

The first transvectant of the binary cubic and its second transvectant is the *cubicovariant* of the binary cubic.

NOTES.

Mr. Joseph Larmer, Fellow of St. John's College, Cambridge, has been elected Lucasian professor of mathematics to succeed the late Sir George Gabriel Stokes. D.

Dr. Arnold Emch, heretofore Assistant Professor of Pure and Applied Mathematics in the University of Colorado, has been promoted by the Regents to

a full Professorship of Graphics and Mathematics. Dr. Emch first came to the University in the spring of 1900. The following fall he was made an Assistant in Mathematics, and before the year closed he was elected Assistant Professor of Pure and Applied Mathematics. His recent promotion to a full Professorship brings Dr. Emch next to Professor DeLong in the Mathematical Department, and is a well-deserved recognition of his ability, faithfulness, and teaching power.

BOOKS.

Plane Geometry by the Suggestive Method. By John A. Avery, Head of the Mathematics Department, English High School, Somerville, Mass. 8vo. Board, vi+122 pages. Boston: Benj. H. Sanborn & Co.

In this work, the demonstrations are not given in full, but are outlined by means of hints or suggestions. No figures are given but space is left under each theorem so that the pupil can fill in the figure for himself.

The book should prove valuable in the hands of teachers having time to give personal attention to the work of each pupil. F.

A Text-Book of Field Astronomy for Engineers. By George C. Comstock, Director of the Washburn Observatory, Professor of Astronomy in the University of Wisconsin. 8vo. Cloth, x+202 pages. Illustrated. Price $2.50. New York: John Wiley & Sons.

This work is designed chiefly for students in technical schools, where the work in Astronomy is usually a part of a course of technical and professional training of students who have no purpose to become astronomers. Such students are often ignorant of many important problems in Astronomy which are of great value to them in practical life. In this work are considered such problems which, in the opinion of the author, are most useful, among them being:

Rough Determinations of Latitude from altitude; Time and azimuth from single altitude; approximate Determinations circum-meridian altitudes for altitude; Time for single-altitude, etc. F.

The Constructive Development of Group Theoy. By B. S. Easton. Publications of the University of Pennsylvania, Mathematics, No. 2. iv+89 pages. Cloth. Price, 75 cents.

This monograph presents in continuous form, but omitting all proofs, the main concepts and results of abstract and substitution group theory. While the theory of linear groups is expressly excluded, some of its results are given under "systems of simple groups," pp. 83–84. An exhaustive bibliography occupies only 34 pages. The treatise proper occupies 39 pages, exclusive of the 8 pages of tables. A systematic use of abbreviations for titles and journals has enabled the author to give a vast number of references in so short a space. The monograph will appeal both to the beginner and to the specialist in group-theory. A technical review, noting some minor corrections, has been offered to the Bulletin of the American Mathematical Society. D.

A Mathematical Solution Book. Fourth Edition, revised and enlarged. By B. F. Finkel, Professor of Mathematics and Physics in Drury College. Cloth, xvi+549 pages. Price, $2.00. Springfield: Kibler & Co.

The need for such a text in English is evinced by its cordial reception, the book having passed through four editions in a comparatively short time. It aims to give "a solution of every problem presenting anything peculiar and of those problems which go the rounds of the conntry." While devoted chiefly to the essentials of arithmetic, algebra and geometry, it gives an abundance of interesting material not usually accessible to the teacher or student of elementary mathematics, and hence should prove a stimulus towards a wider knowledge of these subjects, It is aimed to supplement the usual texts. Occasionally a higher subject is drawn upon in the proofs; circulating decimals are treated by use of the algebraic theory of infinite geometrical progressions; the formulae in interest and annuities are obtained by algebraic methods; many of the formula in mensuration (after the 25 pages of elementary treatment) are established with the aid of Calculus. In the definitions of mathematical terms, the Greek and Latin derivations are given. The subject of higher plane curves receives considerable attention. As an exception to the usual care in the matter of definitions, that of a regular solid (p. 343) seems indaequate. The aggregate of two regular tetrahedrons, forming a dihedron, has its faces all equal, but is not a regular solid. Again, tetrahedron, octahedron, etc., are used by the author to denote regular solids only. As an introduction to geometry, a chapter on logic and logical fallacies is aptly inserted. The book surely merits the success it is receiving. D.

Niedere Zahlentheorie. Part I. By Dr. Paul Bachmann. B. G. Teubner, Liepzig. 1902. x + 402 pages.

This is the first of two volumes on elementary number theory written for Teubner's extensive collection of text-books on higher mathematics. It doesn't conflict seriously with Bachmann's ambitious series of volumes giving a comprehensive exposition of the theory of numbers, but rather forms a supplementary volume. An interesting feature is the exhaustive treatment of the known proofs by elementary number theory of the quadratic reciprocity law and the inter-relations of these proofs. While it is not as well adapted for a brief introduction to number theory as many of the current texts, it will appeal to the student wishing a complete account of the results and methods of the elementary parts of the theory of numbers. D.

Liniengeometrie mit Anwendungen. Volume I. By Dr. Konrad Zindler, Professor in the University of Innsbruck. Liepzig: G. T. Görshen, 1902. 8vo. viii + 380 pages. 87 figures.

This text-book, containing numerous exercises with hints for their solution, forms volume 34 of Schubert's collection of texts. It is a systematic exposition of Linegeometry, employing both synthetic and analytic methods. The present (first) volume treats chiefly of linear complexes and congruences of lines, together with applications to the theory of motion, mechanics, and graphic statics. The second volume will treat algebraic line configurations of higher than the first degree.

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ON THE PRODUCT OF AN ALTERNANT BY A SYMMETRIC FUNCTION.

By DR. W. E. TAYLOR, Associate Professor of Mathematics. Syracuse University.

As is well known, the product of the simple alternant $|\, a_1^0\ a_2^1\ a_3^2\ a_4^3 \ldots .\, |$ and a symmetric function of a_1, a_2, a_3, ... is an aggregate of alternants. When the symmetric function is of the form $\Sigma\, a_1 a_2 a_3 \ldots\ldots a_r$, the product is a single alternant differing from the original in having each of the last exponents increased by unity. In the general case, the mode of obtaining the aggregate is a fairly simple problem. The problem of obtaining the coefficient of a given alternant without having at the same time to obtain the coefficients of all the alternants in the aggregate is a problem not so simple. As a partial solution of this problem Dr. Muir has shown how the coefficient of one term of the aggregate may be obtained independently of the others.

The object of this paper is to extend the general problem started by Muir.

It is apparent that if we have a table giving the coefficients of all alternants in the product, (1)

$$|\, 0123\ldots\ldots(n-1)\, |\, \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_2}\right)^{i_2} \ldots\ldots \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_i}\right)^{i_h}$$

where $0 \leqq i_k \leqq n$, $1 \leqq j_h \leqq n$; $i_1 j_1 + i_2 j_2 + \ldots\ldots + i_h j_h = t \leqq n$; and where $i_1 + i_2 + i_3 + \ldots\ldots i_h = k$, we can by taking the proper multiples* of

$$(2)\quad \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_2}\right)^{i_2} \ldots\ldots \left(\sum a_1 a_2 a_3 \ldots\ldots a_{j_h}\right)^{i_h}$$

form any symmetric function S, and hence from such a table get the coefficients of any alternant in the product $|\, 0\ 1\ 2\ 3 \ldots .(n-1)\, |\, S$.

*These multiples are found from the table of symmetric functions of weight t.

The difference equation for the coefficients of all alternants in the product (1) (where $i_1=1$, $j_1 \overline{\gtrless} \frac{1}{2}n$) is obtained and this is used to construct tables for certain cases.

1. Let $|\, 0\ 1\ 2\ 3 \ldots r_1 s_1 \ldots r_2 s_2 \ldots r_g s_g \ldots r_k s_k \,|$ denote the alternant wherein the numbers are consecutive except at the k points $r_1 s_1$, $r_2 s_2$, ..., $r_k s_k$,—that is, the numbers from 0 to r_1, s_1 to r_2, s_2 to r_3, and so on are consecutive; but $r_1 s_1$, $r_2 s_2$, etc., may differ by more than unity.

Let us denote the coefficient of this alternant in the product

$$|\, 0123 \ldots (n-1) \,| \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)\left(\sum a_1 a_2 a_2 \ldots a_{j_2}\right)^{i_2} \ldots \left(\sum a_1 a_2 a_3 \ldots a_{j_h}\right)^{i_h}$$

by

(3) $$C^t \begin{Bmatrix} s_1 & s_2 & s_3 \ldots s_g \ldots s_k \\ 0 & 0 & 0 \ldots 0 \ldots 0 \end{Bmatrix}$$

and the coefficient of

$$|\, 0\ 1\ 2\ 3 \ldots r_1 (s_1-1) \ldots (s_1+\alpha-1)(s_1+\alpha) \ldots r_2 (s_2-1) s_2 \ldots (s_2+\beta-2)(s_2+\beta) \ldots$$
$$\ldots r_g (s_g-1) s_g \ldots (s_g+\gamma-2) \ldots r_k (s_k-1) \,| \text{ in the product}$$

$$|\, 0123 \ldots (n-1) \,| \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)\left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_h}\right)^{i_h-1}$$

by

$$C^{t-j_h} \begin{Bmatrix} s_1 & s_2 \ldots s_g \ldots s_k \\ \alpha & \beta \ldots \gamma \ldots \kappa \end{Bmatrix},$$

where γ placed below s_g denotes that γ consecutive numbers beginning with s_g are decreased by unity..

2. Now it is easily seen that that the product

$$|\, 0123 \ldots (n-1) \,| \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)\left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_h}\right)^{i_h-1}$$

$$= \ldots + \sum_0^{j_h} \alpha\beta \ldots \kappa \; C^{t-j_h} \begin{Bmatrix} s_1 & s_2 \ldots s_k \\ \alpha & \beta \ldots \kappa \end{Bmatrix} \;|\, 0\ 1\ 2\ 3 \ldots$$

$$\ldots r_1 (s_1-1) s_1 (s_1+1) \ldots (s_1+\alpha-2)(s_1+\alpha) \ldots r_g (s_g-1) s_g \ldots (s_g+\gamma-2)(s_g+\gamma) \ldots$$

$\ldots r_k(s_k-1) \,|$ + other terms, where $\alpha+\beta+\ldots+\kappa=j_h$; κ cannot of course be greater than 1.

Only those terms are written which on multiplying both sides by $\Sigma a_1 a_2 a_3 \ldots a_{j_h}$ can give rise to

$$|\, 0\ 1\ 2\ 3 \ldots r_1 s_1 \ldots r_2 s_2 \ldots r_g s_g \ldots r_k s_k \,| \;;$$

but the coefficient of this term in the product is (3). Hence this coefficient must be equal to the sum of the coefficients of the above terms, that is

$$C^{t}\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ 0 & 0 \ \ldots\ldots 0 \end{Bmatrix} = \sum_{0}^{j_h} {}_{\alpha\,\beta\ldots\ldots\kappa}\, C^{t-j_h}\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ \alpha & \beta \ \ldots\ldots \kappa \end{Bmatrix}.$$

This is the relation between coefficients in the process of multiplication.

3. If we have a table giving the product of $|\,0\ 1\ 2\ 3 \ldots\ldots (n-1)\,|$ by all the symmetric functions of weight w, of the form (2), we can by means of the relation between coefficients found in article 2 construct a table giving the product of $|\,0\ 1\ 2\ 3 \ldots\ldots (n-1)\,|$ by all symmetric functions of the same type of weight $w+j_h$.

The following tables of this kind from weight one to weight seven have been constructed by means of the relation of article 2.

In this way we can construct the table of order t having those of order less than t.

	$\lvert 0123\cdot(n-2)n\rvert$	$\lvert 0123\cdots(n-3)(n-1)n\rvert$	$\lvert 0123\cdots(n-2)(n+1)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)n\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+1)\rvert$	$\lvert 0123\cdot(n-2)(n+2)\rvert$	$\lvert 0123\cdot(n-3)n(n+1)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)(n-1)n\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)(n+1)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-2)(n+3)\rvert$	$\lvert 0123\cdots(n-3)n(n+2)\rvert$	$\lvert 0123\cdots(n-4)(n-2)n(n+1)\rvert$	$\lvert 0123\cdots(n-6)(n-4)\cdots(n-1)n\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)(n-1)(n+1)\rvert$	$\lvert 0123\cdot(n-4)(n-2)(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+3)\rvert$	$\lvert 0123\cdots(n-2)(n+4)\rvert$
Σa_1	1																	
$(\Sigma a_1)^2$		1	1															
$\Sigma a_1 a_2$		1																
$(\Sigma a_1)^3$				1	2	1												
$\Sigma a_1 a_2 \Sigma a_1$				1	1													
$\Sigma a_1 a_2 a_3$				1														
$(\Sigma a_1)^4$							2	1	3	3	1							
$\Sigma a_1 a_2 (\Sigma a_1)^2$							1	1	2	1								
$(\Sigma a_1 a_2)^2$							1	1	1									
$\Sigma a_1 a_2 a_3 \Sigma a_1$								1	1									
$\Sigma a_1 a_2 a_3 a_4$								1										
$(\Sigma a_1)^5$												5	5	1	4	6	4	1
$\Sigma a_1 a_2 (\Sigma a_1)^3$												2	3	1	3	3	1	
$(\Sigma a_1 a_2)^2 \Sigma a_1$												1	2	1	2	1		
$\Sigma a_1 a_2 a_3 (\Sigma a_1)^2$													1	1	2	1		
$\Sigma a_1 a_2 a_3 \Sigma a_1 a_2$													1	1	1			
$\Sigma a_1 a_2 a_3 a_4 \Sigma a_1$														1	1			
$\Sigma a_1 a_2 a_3 a_4 a_5$														1				

	$\lvert 0123\cdots(n-4)(n-1)n(n+1)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)n(n+1)\rvert$	$\lvert 0123\cdots(n-4)(n-2)n(n+2)\rvert$	$\lvert 0123\cdots(n-3)n(n+3)\rvert$	$\lvert 0123\cdots(n-3)(n+1)(n+2)\rvert$	$\lvert 0123\cdots(n-7)(n-6)\cdots(n-1)n\rvert$	$\lvert 0123\cdots(n-6)(n-4)\cdots(n-1)(n+1)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)(n+3)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+4)\rvert$	$\lvert 0123\cdots(n-2)(n+5)\rvert$
$(\sum a_1)^6$ a	5	9	16	9	5	1	5	10	10	5	1
$\sum a_1a_2\,(\sum a)^4$	3	6	8	3	2	1	4	6	4	1	
$(\sum a_1a_2)^2(\sum a_1)^2$	2	4	4	1	1	1	3	3	1		
$(\sum a_1a_2)^3$	1	3	2		1	1	2	1			
$\sum a_1a_2a_3(\sum a_1)^3$	1	3	2			1	3	3	1		
$\sum a_1a_2a_3\sum a_1a_2\sum a_1$	1	2	1			1	2	1			
$(\sum a_1a_2a_3)^2$	1	1				1	1				
$\sum a_1a_2a_3a_4\,(\sum a_1)^2$		1				1	2	1			
$\sum a_1a_2a_3a_4\sum a_1a_2$		1				1	1				
$\sum a_1a_2a_3a_4a_5\,\sum a_1$						1	1				
$\sum a_1a_2a_3a_4a_5a_6$						1					

	$\lvert 0123\cdots(n-5)(n-3)(n-1)n(n+1)\rvert$	$\lvert 0123\cdots(n-6)(n-4)(n-3)(n-2)n(n+1)\rvert$	$\lvert 0123\cdots(n-4)(n-1)n(n+2)\rvert$	$\lvert 0123\cdots(n-5)(n-3)(n-2)n(n+2)\rvert$	$\lvert 0123\cdots(n-4)(n-2)n(n+3)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n+1)(n+2)\rvert$	$\lvert 0123\cdots(n-3)n(n+4)\rvert$	$\lvert 0123\cdots(n-3)(n+1)(n+3)\rvert$	$\lvert 0123\cdots(n-8)(n-6)\cdots(n-1)n\rvert$	$\lvert 0123\cdots(n-7)(n-5)\cdots(n-1)(n+1)\rvert$	$\lvert 0123\cdots(n-6)(n-4)\cdots(n-1)(n+2)\rvert$	$\lvert 0123\cdots(n-5)(n-3)\cdots(n-1)(n+3)\rvert$	$\lvert 0123\cdots(n-4)(n-2)(n-1)(n+4)\rvert$	$\lvert 0123\cdots(n-3)(n-1)(n+5)\rvert$	$\lvert 0123\cdots(n-2)(n+6)\rvert$
$(\sum a_1)^7$	14	14	21	35	35	21	14	14	1	6	15	20	15	6	1
$\sum a_1a_2(\sum a_1)^5$	9	10	11	20	15	10	4	5	1	5	10	10	5	1	
$(\sum a_1a_2)^2(\sum a_1)^3$	6	7	6	11	6	5	1	2	1	4	6	4	1		
$(\sum a_1a_2)^3\,\sum a_1$	4	5	3	6	2	3		1	1	3	3	1			
$\sum a_1a_2a_3(\sum a_1)^4$	4	6	3	8	3	2			1	4	6	4	1		
$\sum a_1a_2a_3\sum a_1a_2(\sum a_1)^2$	3	4	2	4	1	1			1	3	3	1			
$\sum a_1a_2a_3(\sum a_1a_2)^2$	2	3	1	2		1			1	2	1				
$(\sum a_1a_2a_3)^2\sum a_1$	2	2	1	1					1	2	1				
$\sum a_1a_2a_3a_4(\sum a_1)^3$	1	3		2					1	3	3	1			
$\sum a_1a_2a_3a_4\sum a_1a_2\sum a_1$	1	2		1					1	2	1				
$\sum a_1a_2a_3a_4\sum a_1a_2a_3$	1	1							1	1					
$\sum a_1a_2a_3a_4a_5(\sum a_1)^2$		1							1	2	1				
$\sum a_1a_2a_3a_4a_5\sum a_1a_2$		1							1	1					
$\sum a_1a_2a_3a_4a_5a_6\sum a_1$									1	1					
$\sum a_1a_2a_3a_4a_5a_6a_7$									1						

4. If we denote the product

$$| \, 0123.....(n-1) \, | \left(\sum a_1 a_2 a_3 .....a_{j_2}\right)^{i_2} \left(\sum a_1 a_2 a_3 .....a_{j_3}\right)^{i_3} ..... \left(\sum a_1 a_2 a_3 .....a_{j_h}\right)^{i_h - 1}$$

by P, then the alternant

$| \, 0123.....r_1 s_1 .....r_2 s_2 .....r_g s_g .....r_k s_k \, |$ —which we shall represent temporarily by A_1— in the product $P\Sigma a_1 a_2 a_3 .....a_{j_1}$, (or P_1), can arise from the following alternants of P:

$$| \, 0\;1\;2\;3.....r_1 (s_1 - 1) s_1 (s_1 + 1) .....(s_1 + \alpha_1 - 2)(s_1 + \alpha_1) .....r_g (s_g - 1)(s_g) .....$$

$$.....(s_g + \gamma_1 - 2)(s_g + \gamma_1) .....r_k (s_k - 1) \, | \, ,$$

where $\alpha_1, \beta_1, ...\gamma_1 .....\kappa_1 = 0, 1, 2, 3.....j_1$,

and $\alpha_1 + \beta_1 + \gamma_1 .....\kappa_1 = j_1$.

The alternant

$$| \, 0\;1\;2\;3.....(r_1 - j_h + 1) .....r_1 (r_1 + 1) s_1 .....r_2 s_2 .....r_g \, s_g \, ...r_k s_k \, | \, ,$$

which we shall also denote temporarily by A_2 in the product $P.\Sigma a_1 a_2 a_3 .....a_{j_1} + j_h$, (or P_2), will evidently arise from the same terms of P, but it will arise also from the following terms:

$$| \, 0\;1\;2\;3.....(r_1 - j_h + 1) .....r_1 (r_1 + 1) s_1 (s_1 + 1) .....(s_1 + \alpha + \alpha_1 - 2)(s_1 + \alpha + \alpha_1) .....$$

$$.....r_g \, s_g \, (s_g + 1) .....(s_g + \alpha + \alpha_1 - 2)(s_g + \alpha + \alpha_1) .....r_k (s_k - 1) \, | \, .$$

These terms could not exist, however, if $j_1 \overline{\gtrless} \frac{1}{2} n$.

5. If we denote the coefficient of A_2 by

$$C^{t \; j_h}_{\;\; r_1} \begin{Bmatrix} s_1 & s_2 .....s_k \\ 0 & 0 \; .....0 \end{Bmatrix}$$

where the j_h above the r_1 denotes that r_1 and the preceding $(j_h - 1)$ numbers are increased by unity.

Then, if $j_1 \overline{\gtrless} \frac{1}{2} n$, we will have

$$C^{t \; j_h}_{\;\; r_1} \begin{Bmatrix} s_1 & s_2 .....s_k \\ 0 & 0 \; .....0 \end{Bmatrix} = \sum_0^{j_h} \alpha \, \beta .....\kappa \; C^{t - j_h \; j_h}_{\;\;\;\;\;\; r_1} \begin{Bmatrix} s_1 & s_2 .....s_k \\ \alpha & \beta \; .....\kappa \end{Bmatrix} ;$$

for, under these conditions the coefficient of A_1 in the product P_1 is the same as the coefficient of A_2 in P_2.

This gives us a means of obtaining the coefficient of any alternant in the product $P_1 . \Sigma a_1 a_2 a_3 ..... a_{j_h}$, provided we have all those in the product P_2; that is, we obtain coefficients in this table from others in the same table.

6. Having an incomplete table as is considered in the preceding article we could readily convert it into a complete table of order $(t-j_1)$ by removing in all possible ways j_1 numbers from the alternants in the product.

7. Let us now consider, in the product P_1:

$$| \, 0 \; 1 \; 2 \; 3 ..... (n-1) \, | \sum a_1 a_2 a_3 .... a_{n-i} \left(\sum a_1\right)^i,$$

those alternants of the type

$$| \, 0 \; 1 \; 2 \; 3 ..... r_1 s_1 ..... r_2 s_2 ..... r_3 s_3 \, | \, ,$$

where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, and $s_3 - r_3 = q$.

This alternant in P_1 can evidently arise on multiplying

$$| \, 0 \; 1 \; 2 \; 3 ..... r_1 s_1 .... r_2 s_2 .... r_3 (s_3 - 1) \, |$$
$$| \, 0 \; 1 \; 2 \; 3 ..... r_1 s_1 .... r_2 (s_2 - 1)(s_2 + 1) ..... r_3 s_3 \, |$$
$$| \, 0 \; 1 \; 2 \; 3 ..... r_1 (s_1 - 1)(s_1 + 1) ..... r_2 s_2 .... r_3 s_3 \, |$$

by Σa_1, in the product

$$| \, 0 \; 1 \; 2 \; 3 ..... (n-1) \, | \sum a_1 a_2 a_3 ..... a_{n-i} \left(\sum a_1\right)^{i-1},$$

so that we have

$$C^i \begin{Bmatrix} s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{Bmatrix} = \sum_0^1 {}_{\alpha\beta\gamma} \, C^{i-1} \begin{Bmatrix} s_1 & s_2 & s_3 \\ \alpha & \beta & \gamma \end{Bmatrix} \quad (\alpha + \beta + \gamma = 1).$$

8. Since $n - 1 \overline{\gtrless} \frac{1}{2} n$, the conditions of articles 4 and 5 are satisfied and we have the following reduction formula for the construction of tables:

$$C^{i\,0}_{\;s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix} = C^{i-1\;1}_{\;\;s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-1\;1}_{\;\;s_1} \begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix} + C^{i-1\;0}_{\;\;s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix}$$

In the following table

$$\overline{\sum a_1} = | \, 0 \; 1 \; 2 \; 3 ..... (n-1) \, | \sum a_1 a_2 a_3 ..... a_{n-1} \left(\sum a_1\right)^1,$$

$$\overline{\left(\sum a_1\right)^2} = | \, 0 \; 1 \; 2 \; 3 ..... (n-1) \, | \sum a_1 a_2 a_3 ..... a_{n-2} \left(\sum a_1\right)^2,$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\overline{\left(\sum a_1\right)^{11}} = | \, 0 \; 1 \; 2 \; 3 ..... (n-1) \, | \sum a_1 a_2 a_3 ..... a_{n-11} \left(\sum a_1\right)^{11}.$$

G.P.B	|123.. n|	|023.. $(n-1)(n+1)$|	|013. $(n-2)n(n+1)$|	|013 $(n-1)(n+2)$|	|0124 $(n-3)(n-1)n(n+1)$|	|0124 $(n-2)n(n+2)$|	|0124 $(n-1)(n+3)$|	|01235 $(n-4)(n-2)(n-1)n(n+1)$|	|01235 $(n-3)(n-1)n(n+2)$|	|01235 $(n-2)n(n+3)$|	|01235 $(n-1)(n+4)$|	|01..46 $(n-5)(n-3)\cdots n(n+1)$|	|01..46 $(n-4)(n-2)(n-1)n(n+2)$|	|01..46 $(n-3)(n-1)n(n-3)$|	|01..46 $(n-2)n(n+4)$|	|01..46 $(n-1)(n+5)$|	|01..57 $(n-6)(n-4)\cdots n(n+1)$|	|01..57 $(n-5)(n-3)\cdots n(n+2)$|	|01..57 $(n-4)(n-2)(n-1)n(n+3)$|	|01..57 $(n-3)(n-1)n(n+4)$|	|01..57 $(n-2)n(n+5)$|	|01..57 $(n-1)(n+6)$|	|01..68 $(n-7)(n-5)\cdots n(n+1)$|	|01..68 $(n-6)(n-4)\cdots n(n+2)$|	|01..68 $(n-5)(n-3)\cdots n(n+3)$|	|01..68 $(n-4)(n-2)(n-1)n(n+4)$|	|01..68 $(n-3)(n-1)n(n+5)$|	|01..68 $(n-2)n(n+6)$|	|01..68 $(n-1)(n+2$
$(\overline{\Sigma a_1})^1$	1	1																											
$(\overline{\Sigma a_1})^2$	1	2	1	1																									
$(\overline{\Sigma a_1})^3$	1	3	3	3	1	2	1																						
$(\overline{\Sigma a_1})^4$	1	4	6	6	4	8	4	1	3	3	1																		
$(\overline{\Sigma a_1})^5$	1	5	10	10	10	20	10	5	15	15	5	1	4	6	4	1													
$(\overline{\Sigma a_1})^6$	1	6	15	15	20	40	20	15	45	45	15	6	24	36	24	6	1	5	10	10	5	1							
$(\overline{\Sigma a_1})^7$	1	7	21	21	35	70	35	35	105	105	35	21	84	126	84	21	7	35	70	70	35	7	1	6	15	20	15	6	1
$(\overline{\Sigma a_1})^8$	1	8	28	28	56	112	56	70	210	210	70	56	224	336	224	56	28	140	280	280	140	28	8	48	120	160	120	48	8
$(\overline{\Sigma a_1})^9$	1	9	36	36	84	168	84	126	478	478	126	126	504	756	504	126	84	420	840	840	420	84	36	216	540	720	540	216	36
$(\overline{\Sigma a_1})^{10}$	1	10	45	45	120	240	120	210	630	630	210	252	1008	1512	1008	252	210	1050	2100	2100	1050	210	120	720	1800	2400	1800	720	120
$(\overline{\Sigma a_1})^{11}$	1	11	55	55	165	330	165	330	990	990	330	462	1848	2772	1848	462	462	2310	4620	4620	2310	462	330	1980	4950	6600	4950	1980	330

9. We wish to prove the formula

$$C^{i0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=\frac{i(i-1)\ (i-2)\ldots\ldots(i-s_1+2)}{\underline{|s_1-1}}\cdot\frac{(s_1-2)(s_2-3)\ldots\ldots(s_1-s_3+r_3-2)}{\underline{|s_3-r_3-1}}.$$

In the reduction formula of article 8, let i receive the successive values 2, 3, 4, 5.....i, and we have the following result:

$$C^{2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+C^{1\,1}_{s_1}\left\{\begin{matrix} s_1 & s_2 \\ 1 & 0\end{matrix}\right\}+C^{1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 0 & 0\end{matrix}\right\}$$

$$C^{3\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+C^{2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}+C^{2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}$$

$$C^{4\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{3\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}+C^{3\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}+C^{3\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}$$

$$C^{5\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{4\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+C^{4\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}+C^{4\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}$$

....

$$C^{i-1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{i-2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+C^{i-2\,1}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 1 & 0\end{matrix}\right\}+C^{i-2\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=C^{i-1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+C^{i-1\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}+C^{i-1\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=\sum_{1}^{i-1} j\ C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_2 \\ 0 & 1\end{matrix}\right\}+\sum_{1}^{i-1} j\ C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}$$

10. Let the reduction formula, designated R,

$$C^{i\,0}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 0\end{matrix}\right\}=\sum_{1}^{i-1} j\ C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 0 & 1\end{matrix}\right\}+\sum_{1}^{i-1} j\ C^{j\,1}_{s_1}\left\{\begin{matrix} s_2 & s_3 \\ 1 & 0\end{matrix}\right\}$$

operate upon itself and the successive results:

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix} = C^{i-1\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-1\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$C^{i-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$C^{i-3\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-3\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$C^{s_1-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{s_1-2\,1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix}$$

Observe that the terms of R beyond the limit s_1-2 are zero.

$$=1\left(C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_2 \\ 1 & 1 \end{Bmatrix} + C^{i-2\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$2\left(C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{i-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$(a) \qquad 3\left(C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{i-4\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 2 \end{Bmatrix}\right)$$

$$\cdot \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$(i-2)\ C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 0 \end{Bmatrix} + 2\,C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 1 \end{Bmatrix} + C^{s_1-3\,2}_{s_1}\begin{Bmatrix} s & s_3 \\ 0 & 2 \end{Bmatrix}$$

$$=1\left(C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-3\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$3\left(C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-4\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$(b)\,6\left(C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-5\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$10\left(C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix} + 3\,C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix} + 3\,C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix} + C^{i-6\,3}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 3 \end{Bmatrix}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{(i-2)(i-3)}{\underline{|2}}\Big(C^{s_1-4\,3}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ 3 & 0 \end{Bmatrix}+3C^{s_1-4\,3}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ 2 & 1 \end{Bmatrix}$$

$$+3C^{s_1-4\,3}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ 1 & 2 \end{Bmatrix}+C^{s_1-4\,3}_{\;s_1}\begin{Bmatrix} s_2 & c_3 \\ 0 & 3 \end{Bmatrix}\Big)$$

....

....

$$=1\Big(C^{i-s_1+2\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-2 & 0 \end{Bmatrix}+(s_1-2)\,C^{i-s_1+2\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-3 & 1 \end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\underline{|2}}\,C^{i-s_1+2\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2 \end{Bmatrix}+\ldots\Big)$$

$$(s_1-2)\Big(C^{i-s_1+1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-2 & 0 \end{Bmatrix}+(s_1-2)\,C^{i-s_1+1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-3 & 1 \end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\underline{|2}}\,C^{i-s_1+1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2 \end{Bmatrix}+\ldots\Big)$$

$$(c)\;\frac{(s_1-2)(s_1-3)}{\underline{|2}}\Big(C^{i-s_1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-2 & 0 \end{Bmatrix}+(s_1-2)\,C^{i-s_1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-3 & 1 \end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\underline{|2}}\,C^{i-s_1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2 \end{Bmatrix}+\ldots\Big)$$

$$\frac{(s_1-2)\,(s_1-3)\,(s_1-4)}{\underline{|3}}\Big(C^{i-s_2-1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-2 & 0 \end{Bmatrix}$$

$$+(s_1-2)\,C^{i-s_1-1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-3 & 1 \end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\underline{|2}}\,C^{i-s_2-1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2 \end{Bmatrix}+\ldots\Big)$$

....

$$\frac{(i-2)\,(i-3)\ldots(i-s_1+2)}{\underline{|s_1-3}}\Big(C^{1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-2 & 0 \end{Bmatrix}+(s_1-2)\,C^{1\,s_1-1}_{\;s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-3 & 1 \end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}C^{1}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & s_3\\ s_1-4 & 2\end{matrix}\right\}+\ldots\ldots+\ldots\ldots$$

$$\ldots\ldots+\frac{(s_1-2)(s_1-3)\ldots\ldots(s_1-k-1)}{\lfloor k}C^{1}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & s_3\\ s_1-k-2 & k\end{matrix}\right\}+\ldots\ldots\Big).$$

In (c) let $i=i$, $s_1=s_1$, and $s_3=r_3+1$. Then

$$C^{i}{}_{s_1}^{0}\left\{\begin{matrix}s_2 & (r_3+1)\\ 0 & 0\end{matrix}\right\}=1C^{i-s_1+2}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & (r_3+1)\\ s_1-2 & 0\end{matrix}\right\}+$$

$$(s_1-2)C^{i-s_2+1}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & (r_3+1)\\ s_1-2 & 0\end{matrix}\right\}+\frac{(s_1-1)(s_1-2)}{\lfloor 2}C^{i-s_1}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & (r_3+1)\\ s_2-1 & 0\end{matrix}\right\}$$

$$\ldots\ldots+\frac{(i-2)\,(i-3)(i-4)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}C^{1}{}_{s_1}^{s_1-1}\left\{\begin{matrix}s_2 & (r_3+1)\\ s_1-1 & 0\end{matrix}\right\}.$$

It is evident from the table that the C's have values, respectively, from $i-s_1+2$ down to 1. Wherefore

$$C^{i}{}_{s_1}^{0}\left\{\begin{matrix}s_2 & (r_3+1)\\ 0 & 0\end{matrix}\right\}=1.(i-s_1+2)+(s_1-2)(i-s_1+1)+\frac{(s_1-1)(s_1-2)}{\lfloor 2}(i-s_1)$$

$$+\frac{s_1(s_1-1)(s_1-2)}{\lfloor 3}(i-s_1-1)+\ldots\ldots+\frac{(i-2)(i-3)(i-4)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}(i-(i-1))$$

$$=i\Big[1+(s_1-2)+\frac{(s_1-1)(s_1-2)}{\lfloor 2}+\frac{s_1(s_1-1)(s_1-2)}{\lfloor 3}+$$

$$\ldots\ldots+\frac{(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}\Big]-(s_1-2)\Big[1+(s_1-1)$$

$$+\frac{(s_1-1)s_1}{\lfloor 2}+\frac{(s_1-1)s_1(s_1+1)}{\lfloor 3}+\ldots\ldots+\frac{(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1+2}\Big].$$

Summing the two series in the brackets, we have

$$C^{i}{}_{s_1}^{0}\left\{\begin{matrix}s_2 & (r_3+1)\\ 0 & 0\end{matrix}\right\}=\frac{i(i-1)\,(i-2)\ldots\ldots(i-s_1+2)}{\lfloor s_1-2}-(s_1-2)\frac{i(i-1)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}$$

$$=\frac{i(i-1)(i-2)(i-3).....(i-s_1+2)}{|\underline{s_1-1}}(s_1-1-(s_1-2))$$

$$=\frac{i(i-1)(i-2)(i-3).....(i-s_1+2)}{|\underline{s_1-1}}.1.$$

In a similar manner, it can be proven that

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & r_3+2 \\ 0 & 0 \end{Bmatrix}=\frac{i(i-1)(i-2)(i-3).....(i-s_1+2)}{|\underline{s_1-1}}(s_1-2)$$

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & r_3+3 \\ 0 & 0 \end{Bmatrix}=\frac{i(i-1)(i-2)(i-3).....(i-s_1+2)}{|\underline{s_1-1}}\frac{(s_1-2)\ (s_1-3)}{|\underline{2}},$$

or, in general,

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix}=\frac{i(i-1)(i-2)(i-3).....(i-s_1+2)}{|\underline{s_1-1}}$$

$$\times\frac{(s_1-2)\ (s_1-3).....(s_1-s_3+r_3-2)}{|\underline{s_3-r_3-1}}.$$

SYRACUSE UNIVERSITY, *December 18, 1902.*

AN ACCOUNT OF PROFESSOR RUNKLE'S MATHEMATICAL MONTHLY.

By PROFESSOR SIMON NEWCOMB.

I first made Mr. Runkle's acquaintance in the winter of 1857, when he was senior assistant in the Nautical Almanac Office, then at Cambridge, Mass. His intelligence, intellectual activity, and lively interest in matters and things generally, not excluding things political, made him a very interesting character.

It was early in 1858 that he announced to me and some others in the office his intention of starting a mathematical journal. His first step was to secure the necessary support. It may be feared that few in our day have an adequate conception of the backward condition of mathematical study in our country at that time. A curious illustration is offered by Davies' well-known dictionary of mathematics, in the preface of which it was announced that it contained defin-

itions of all the terms used in mathematics. The word "determinant" does not, however, occur in the dictionary, nor any terms used in the branches of algebra with which determinants would be associated. So far as I know, the first person in our country to know that a determinant existed, or to become acquainted with a paper in any European mathematical journal, was Prof. Benj. Peirce. Cayley and Sylvester were then at the height of their activity in developing the newer theories, but Peirce was the only one among us who had the slightest idea what they were working at.

Such was the field in which Runkle proposed to sow the seeds of what might be considered a new branch of learning. His circulars were sent to all the leading mathematical professors in the country and others interested in the subject. The responses were such as to justify his going on with the undertaking, although the problem of making a respectable standard compatible with the necessary pecuniary support was a very doubtful one.

The first number was issued in January, 1859, and showed an attempt to compromise between the two difficulties. Its most marked feature, and one which seemed admirably adapted to make it conducive to the advancement of mathematical science, was the prize problem for students. These were of various degrees of simplicity, though all were above the school-boy standard. A first prize was offered for the best solution, and a second prize for the next best. Prize essays were also invited. One of the problems of the first number, which related to a theorem regarding the intersection of circles, brought out a short paper from Cayley, in which he extended the theorem to conics.

The Memoir which won the first prize was by G. W. Hill, on a subject relating to the theory of the figure of the earth. It was almost his first paper published, but showed unmistakably the hand of the master.

The prizes were awarded on the recommendation of a committee to whom the solutions were submitted. Professor Winlock was the first chairman of the committee, but soon retired from it, and the solutions were afterwards passed upon by W. P. G. Bartlett and myself.

Bartlett was a man whose untimely death was a great loss for mathematical science in our country. He was not only of the first order of ability, but an enthusiastic devotee of mathematical study, whose character and standard were in every respect the highest. He was a nephew of the eminent lawyer, Mr. Sidney Bartlett, long a leader of the Boston bar. One day when his uncle urged him to undertake the profession of law in order to secure wealth and position in society, he replied: "If I can secure a high position in the regard of Pierce, Cayley, Sylvester, and fifteen or twenty other men like them, I shall value it more than any opinion that society at large can form of me."

Although not formally connected with the responsible editorship, Bartlett and myself naturally acted as general advisers and referees on matters connected with the Journal.

Among our amusing experiences was one with a Pennsylvania politician of prominence in the State, who had written and read before a Teachers' Associa-

tion a long and rambling essay on the then celebrated problem of the oxen and the field of grass. Not knowing what to do with the paper, the Association referred it to the editor of the Mathematical Monthly, by whom it was consigned to the waste basket. This was a great humiliation to the honorable gentleman, who, from the high office which he held in Washington, opened up a correspondence with the editor. The outcome of it was a great reduction in its length by the author and a subsequent abbreviation by the editor, which reduced it to four pages, and then it was printed, much to Bartlett's disgust.

On another occasion a professor somewhere in the South sent a paper, much of which was extracted, almost verbatim so far as the formulæ went, from Walton's Problems of Mechanics. The question arose how to let the man know, in the gentlest way, that we had detected the fraud. Bartlett's advice was characteristic: "Just write to him that we don't publish stolen articles." I suggested that it would suffice to let him understand in the gentlest way that we knew where the matter came from. This Runkle did, calling his attention to the similarity of his paper to the book of Walton, with which the professor was probably not acquainted. The answer showed that we had mistaken our man. He assured us that he had a copy of Walton and knew all about the matter, and did not think the resemblance was any objection to his paper. There was an added intimation that, while he should be glad to write for the Journal, he had no time to spend in writing rejected articles.

It was soon found that the conduct of the Journal was beset with even greater difficulties than those of dealing with would-be contributors. Unsustained by any public body, it needed a good list of subscribers to be self-supporting. Between the highest grade of mathematics that could be understood by a sufficient number of professors and students in our country at that time, and the lowest that would be of interest to the professional mathematician, there was a very wide gap. In trying to fill it, complaints were heard on the one side that the papers were too difficult to be understood, and, on the other side, that the standard was too low for a journal which was really to promote the advancement of mathematics.

Yet another question was whether it would be allowable to go outside its proper field for the sake of enlarging its sphere of readers. In one of its early numbers was published a paper by Geo. P. Bond on Donati's Comet which, being the first scientific description of that brilliant object to which the public had access, did a great deal to excite interest in the Journal. It thus lost the support of no less a personage than Professor Peirce, who held that the Journal had no right to go outside its proper field.

At the end of the year 1860 three annual volumes had been published. The Civil War was then imminent and public conditions such that the thought of continuing the enterprise had to be abandoned. The historic value of the publication, I think, can not be doubted. Its contents were more varied and interesting, and low though its standard might have been when we measure it by that of the foreign journals, in contained more varied matter of interest to the

student, even of today, than any American mathematical journal that had preceded it. Its great mission, which it successfully fulfilled, was to interest our college men in the subject. Although we do not find among its contributors the names of any who, in recent times, have become prominent as mathematical investigators, I believe that the stimulus it gave to the study of the subject may have been an important factor in the serious advance which commenced fifteen years later at the Johns Hopkins and other of our great universities.

THE AYER PAPYRUS.*

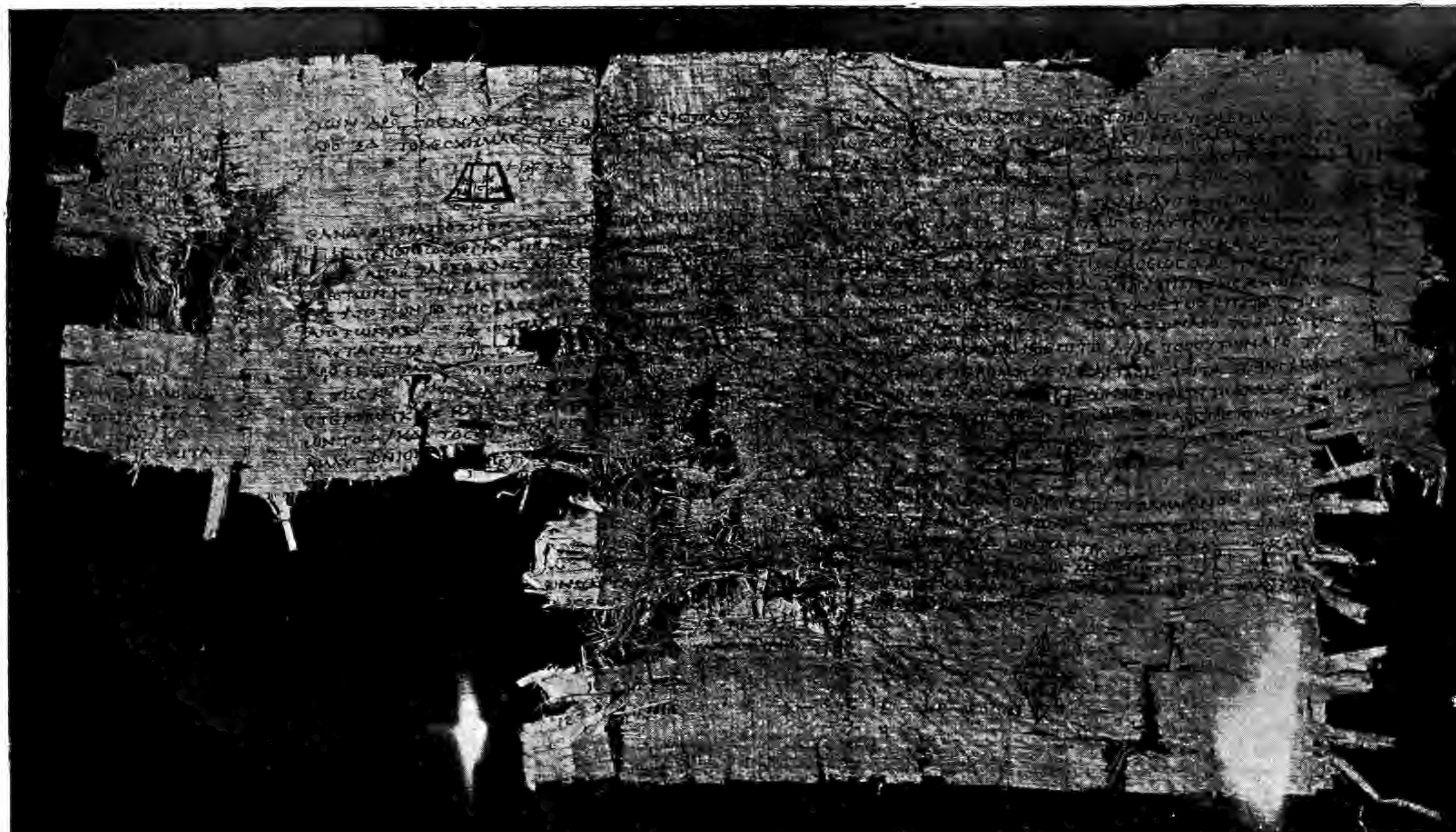
Edited by EDGAR J. GOODSPEED.

In the Eygptian Room of the Field Columbian Museum, Chicago, there is a Greek papyrus of much mathematical interest. It preserves a series of geometrical processes, in which the areas of various quadrilaterals are determined by methods much less advanced than those of Euclid. The papyrus measures cm. 21.3 x 40.5. Originally it formed part of a roll, written in clear uncial characters on one side only, the writing being in columns. Parts of three columns remain, the second and third being nearly complete, while of the first there are but a few scattered words. The uniformity of the language used, and the preservation of three out of the four figures illustrating the processes of these two columns make their restoration easy and certain. The papyrus comes from the Fayûm, and was brought from Egypt to the Field Museum by Mr. Ed. E. Ayer, of Chicago, in 1895. It belongs to the first century after Christ, but the methods pursued in the calculations, and the senses in which certain terms are used carry back the probable date of the work into pre-christian times and suggest relationship with the school of Heron of Alexandria. The work was apparently a practical treatise on mensuration, designed for use in re-surveying farm-lands of irregular shape, after the annual inundation.

Geometrical figures illustrating the proceses described are appended to them. These figures are covered by numerals indicative of the length of each side, part of a side, and perpendicular, and of the area of each section. It will be observed that in the papyrus these figures are very roughly drawn, and by no means satisfy the proportions thus prescribed for them. For the restoration of the second figure, which is missing from the papyrus through mutilation at that point, I am indebted to Professor E. H. Moore. It should be noted that the word translated "acre" is the Greek "aroura," which is strictly equivalent to about two-thirds of the English acre.

The processes, literally translated, are as follows:

*For the Greek text see Edgar J. Goodspeed, *American Journal of Philology*, Vol. XIX, pp. 25-39.



THE AYER PAPYRUS.

From the *American Journal of Philology*, Vol. X.

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The processes, literally translated, are as follows:

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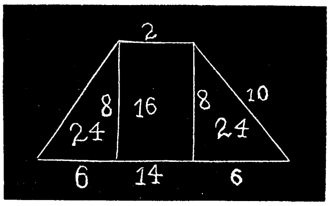
Column I (restored).

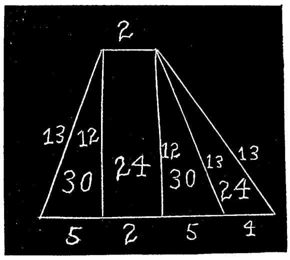
[If there be given an isosceles trapezoid such as the one drawn below, according to the conditions of the problem, the 10 squared=100, and the 2 of the upper side from the 14 of the base leaves 12, $\frac{1}{2}$ of which is 6. This squared=36. Subtract this from 100; the remainder is 64, of which the square root is 8, which is the length of the perpendicular. $\frac{1}{2}$ of this=4, and this by the 6 of the base=24; of so many acres is each of the right-angled triangles. And the 8 of the perpendicular by the 2 of the base=16; of so many]

Column II.

acres is the rectangle in it. Altogether, 64 acres. And the figure will be as follows.



If there be given a scalene trapezoid such as the one drawn below, according to the conditions of the problem, the 13 squared=169, and the 15 squared=225. Subtract the 169; the remainder is 56. Subtract the 2 of the upper side from the 16 of the base; the remainder is 14. Take $\frac{1}{14}$ of 56; it is 4. This from the 14 of the base leaves 10. $\frac{1}{2}$ of this leaves 10. $\frac{1}{2}$ of this=5. This squared=25. Subtract this from the 169; the remainder is 144, of which the square root is 12, which is the length of the perpendicular. This by the 5 of the base=60, $\frac{1}{2}$ of which is 30; of so many acres is each of the right-angled triangles. And the 12 (multiplied) by the 2 of the upper side=24; of so many acres is the rectangle in it. And the 12 multiplied by the 4 of the base=48, $\frac{1}{2}$ of which is 24; of so many acres is the obtuse-angled triangle in it. Altogether, 108 acres. And the figure will be as follows.

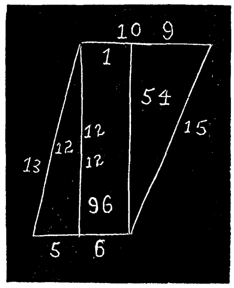


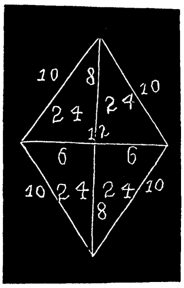
[Figure restored].

Column III.

If there be given a perallelogram such as the one drawn below, according to the conditions of the problem, the 13 of the side squared=169, and the 15 of the side squared=225. Subtract 169 from this; the remainder is 56. Subtract the 6 of the base from the 10 of the upper side; the remainder is 4. Take $\frac{1}{4}$ of 56; it is 14. Subtract the 4; the remainder is 10, $\frac{1}{2}$ of which is 5, which is the length of the base of the right-angled triangle. This squared=25. And the 13 squared=169. Subtract the 25; the remainder is 144, the square root of which is 12, which is the length of the perpendicular. And subtract the 5 from the 6

of the base; the remainder is one. The 1 from the 10 of the upper side leaves 9, which is the length of the remainder of the upper base, (which remainder is the base) of the right-angled triangle. And the 12 of the perpendicular by the 5 of the base=60, $\frac{1}{2}$ of which is 30; of so many acres is the right-angled triangle in it. And the 12 by the 1=12; of so many acres is the rectangle in it. And 12 by the 9 of the base=108, $\frac{1}{2}$ of which is 54; of so many acres is the other right-angled triangle. Altogether, 96 acres. And the figure will be as follows:





If there be given a rhomb such as the one drawn below, according to the conditions of the problem, the 10 squared=100, and $\frac{1}{2}$ of the 12 of the base is 6. This squared=36. Subtract this (from 100); the remainder is 64, of which the square root is 8, which is the length of the perpendicular. This by the (6) of the base=48, $\frac{1}{2}$ of which is 24; of so many acres is each of the right-angled triangles. Altogether, 96 acres.

And the figure will be as follows.

NOTE.—In order to economize space, the figures are set in the paragraphs to which they belong instead of following them. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A teacher's monthly salary after $m=2$ increases of $p=20$ and $q=10\%$, is $\$M=\$120$. What was the original salary?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and G. W. GREENWOOD, A. B., McKendree College, Lebanon, Ill.

$$\text{Original salary}=\$M/[(1+p)(1+g)\text{.....to } m \text{ terms})$$
$$=\$120/[(1.10)(1.20)]$$
$$=\$120\times\tfrac{10}{11}\times\tfrac{5}{6}=\$90\tfrac{10}{11}.$$

ALGEBRA.

173. Proposed by JOHN M. COLAW. A. M., Monterey, Va.

Solve $\sqrt{(a+x+y)}=x$.....(1), $\sqrt{(b+y+z)}=x$.....(2), $\sqrt{(c+z+x)}=x$.....(3).

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

From (2), (3), and (1), we have

$$x^2-y-z=b.....(4),$$
$$y^2-z-x=c.....(5),$$
$$z^2-x-y=a.....(6).$$

From (4) and (5); from (5) and (6); from (6) and (4), we have

$$(x-y)(x+y+1)=b-c.....(7),$$
$$(y-z)(y+z+1)=c-a.....(8),$$
$$(z-x)(z+x+1)=a-b.....(9).$$

By addition of corresponding members of (7), (8), (9), we get

$$(x-y)(x+y+1)+(y-z)(y+z+1)+(z-x)(z+x+1)=0.....(10).$$

It is easy to see that either $x=y=z$.....(11),
or $x+y+1=y+z+1=z+x+1=0$.....(12),

will satisfy (10). From (11), (4), (5), (6), we find

$$x=y=z=1\pm\sqrt{(b+1)},\ 1\pm\sqrt{(c+1)},\ 1\pm\sqrt{(a+1)}.$$

From (4), (5), (6), (12), we obtain

$$x=\pm\sqrt{(b-1)},\ y=\pm\sqrt{(c-1)},\ z=\pm\sqrt{(a-1)}.$$

Also solved by *MARCUS BAKER*, and *G. B. M. ZERR.*

174. Proposed by HARRY S. VANDIVER, Bala, Pa.

If the quantity x be expressed in the form of a continued fraction P_n/Q_n denoting the $(n+1)$th convergent, with x_n the corresponding complete quotient, then $\dfrac{P_{n-(k-1)}-Q_{n-(k+1)}x}{P_n-Q_nx}=(-1)^{k+1}x_n\times x_{n-1}.....x_{n-k}$.

Solution by G. B. M. ZERR, A. M., Ph.D., The Temple College, Philadelphia, Pa., and J. E. SANDERS, Hackney, Ohio.

$$x=\frac{x_nP_n+P_{n-1}}{x_nQ_n+Q_{n-1}} \text{ or } x_n=\frac{xQ_{n-1}-P_{n-1}}{P_n-xQ_n}$$

Similarly,

$$x_{n-1}=\frac{xQ_{n-2}-P_{n-2}}{P_{n-1}-xQ_{n-1}},\ x_{n-2}=\frac{xQ_{n-3}-P_{n-3}}{P_{n-2}-xQ_{n-2}},\cdots\ x_{n-k}=\frac{xQ_{n-(k+1)}-P_{n-(k+1)}}{P_{n-k}-xQ_{n-k}}.$$

$$\therefore\ x_n x_{n-1}\cdots x_{n-k}=\left[\frac{xQ_{n-1}-P_{n-1}}{P_n-xQ_n}\right].\left[\frac{xQ_{n-2}-P_{n-2}}{P_{n-1}-xQ_{n-1}}\right].\left[\frac{xQ_{n-3}-P_{n-3}}{P_{n-2}-xQ_{n-2}}\right].$$

$$\left[\frac{xQ_{n-4}-P_{n-4}}{P_{n-3}-xQ_{n-3}}\right]\cdots\cdots\left[\frac{xQ_{n-(k+1)}-P_{n-(k+1)}}{P_{n+k}-xQ_{n-k}}\right]$$

$$=(-1)^{k+1}\left[\frac{P_{n-(k+1)}-xQ_{n-(k+1)}}{P_n-xQ_n}\right]=(-1)^{k+1}(A),\text{ suppose.}$$

$$\therefore\ (-1)^{k+1}x_n\times x_{n-1}\cdots x_{n-k}=(-1)^{2k+2}(A)=A=\text{result stated.}$$

Also solved in the same manner by *G. W. GREENWOOD*.

GEOMETRY.

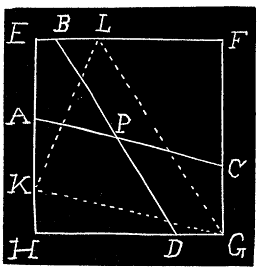
195. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

The diagonals of a four-sided figure are h and k, and the area is A; show that the area of the circumscribing square is

$$\frac{h^2k^2-4A^2}{h^2+k^2-A}.$$

Solution by J. K. HITT, Principal Liberty High School, Goss, Miss.; J. SCHEFFER, A. M., Hagerstown, Md.; and L. L. LOCKE, Professor of Mathematics, Adelphos College, Brooklyn, N. Y.

Let $AC=h$, $BD=k$, be the diagonals of the four-sided figure, and $EFGH$ the circumscribing square. Draw GK, GL, parallel to h, k, respectively, and denote HK, FL, by a, b, respectively.



$\triangle\ GKL=\frac{1}{2}GK.GL\sin KGL=\frac{1}{2}hk\sin P=A$. If $x=$ side of square, we have, $\triangle\ GKL=A=x^2+\frac{1}{2}ax-\frac{1}{2}bx-\frac{1}{2}(a+x)(x-b)=\frac{1}{2}(x^2+ab)$.

Hence, $b=1/a(2A-x^2)$.....(1).

Also, $GL^2=k^2=x^2+b^2$, $GK^2=h^2=x^2+a^2$.

Hence, $a^2=h^2-x^2$.....(2), and, by adding, we get $2x^2=h^2+k^2-(a^2+b^2)$.....(3). Substituting in (3) the values of b and a from (1), (2), we have, $2x^2=(h^2+k^2)-(2A-x^2)^2/(h^2-x^2)-(h^2-x^2)$, whence, $x^2(h^2+k^2-4A)=h^2k^2-4A^2$.

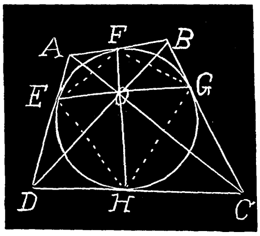
Therefore, $x^2=\dfrac{h^2k^2-4A^2}{h^2+k^2-4A}$.

Also solved by *G. B. M. ZERR*.

196. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a quadrilateral circumscribe a circle, the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

I. Solution by J. R. HITT, Principal, Liberty High School, Goss. Miss.; and G. B. M. ZERR. A. M., Ph. D., The Temple College, Philadelphia, Pa.



Let the diagonals of $ABCD$ meet in O, and let EG, FH meet in O'. FG, EH are the polars of B, D, respectively. Hence, BD is the polar of the intersection of FG, EH. Likewise is AC the polar of intersection of EF, GH. Therefore, O is the pole of third diagonal of $EFGH$. But O' is the pole of the third diagonal of $EFGH$, since the diagonal triangle is self-conjugate with respect to circumscribing circle. Hence O, O', coincide, and the proposition is proved.

II. Solution by G. W. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon. Ill.

We can project the given quadrilateral into a parallelgram circumscribing an ellipse, in which it is easily seen that the lines joining the points of contact of the opposite sides, and the diagonals of the parallelogram all meet in the center, and that these lines in the original figure are concurrent.

Also solved by *L. C. WALKER*, and *L. L. LOCKE*.

CALCULUS.

161. Proposed by J. C. NAGLE. M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

A cylindrical oil tank of length l and radius r is capped by curved ends and rests with the axis horizontal. The total length of the tank is $l+2h$. If the oil stands at depth d in the tank (d less than $2r$) find its volume (a) when the ends are portions of the surface of a sphere, (b) when the ends are portions of the surface of an ellipsoid.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $y^2+z^2=2ry$, be the equation to the cylinder. Then $z=\sqrt{(2ry-y^2)}$.

$$\therefore\ V=2l\int_0^d \sqrt{(2ry-y^2)}dy=\frac{l}{2}\left[\pi r^2+2(d-r)\sqrt{(2rd-d^2)}+2r^2\sin^{-1}\left(\frac{d-r}{r}\right)\right].$$

If $d=2r$, $V=\pi r^2 l$.

(a). Let $x^2+y^2+z^2=R^2=[(r^2+h^2)/2h]^2$, be the equation to a sphere, a portion of whose surface forms the ends of the tank.

$\therefore\ z=\sqrt{(R^2-y^2-x^2)}$, the limits of x are $(R-h)$ and $\sqrt{(R^2-y^2)}$; of y, $-r$ and $d-r$. Let V_1=the volume of both ends.

$$V_1 = 4\int_{-r}^{d-r}\int_{R-h}^{\sqrt{(R^2-y^2)}} \sqrt{(R^2-y^2-x^2)}dydx$$

$$=\int_{-r}^{d-r}\Big[\pi(R^2-y^2)-2(R-h)\sqrt{(2Rh-h^2-y^2)}$$

$$-2(R^2-y^2)\sin^{-1}\Big(\frac{R-h}{\sqrt{(R^2-y^2)}}\Big)\Big]dy.$$

Integrating, substituting the value of R, and reducing,

$$V_1 = \frac{d-r}{12h^2}\Big(3(r^2+h^2)^2-4h^2(d-r)^2\Big)$$

$$\times\Big[\pi-2\sin^{-1}\Big(\frac{r^2-h^2}{\sqrt{[(r^2-h^2)^2+4dh^2(2r-d)]}}\Big)\Big]$$

$$+\tfrac{1}{6}\pi h(3r^2+h^2)-\frac{(d-r)(r^2-h^2)\sqrt{(2rd-d^2)}}{3h}$$

$$+\frac{(r^2+h^2)^3}{6h^3}\tan^{-1}\Big[\frac{(d-r)(r^2-h^2)}{(r^2+h^2)\sqrt{(2rd-d^2)}}\Big]$$

$$-\frac{r^2-h^2}{6h^3}(r^4+4r^2h^2+h^4)\sin^{-1}\Big(\frac{d-r}{r}\Big).$$

If $d=2r$, $V_1=\frac{1}{3}\pi h(3r^2+h^2)$. If $d=2r$, $h=r$, $V_1=\frac{4}{3}\pi r^3$.

Total volume$=V+V_1$.

(*b*). Let $x^2/a^2+(y^2+z^2)/b^2=1$, be the equation to the ellipsoid. It is necessary to know a or b.

If a is known, $b=ar/\sqrt{(2ah-h^2)}$.

$z=b\sqrt{[1-(y^2/b^2)-(x^2/a^2)]}$, the limits of x are $(a-h)$ and $a/b\sqrt{(b^2-y^2)}$; of y, $-r$ and $d-r$.

Let V_2=volume of both ends.

$$V_2=4b\int_{-r}^{d-r}\int_{a-h}^{a/b\sqrt{(b^2-y^2)}}\sqrt{[1-(y^2/b^2)-(x^2/a^2)]}dydx,$$

$$=\int_{-r}^{d-r}\Big[\frac{\pi a}{b}(b^2-y^2)-\frac{2(a-h)}{a}\sqrt{[b^2(2ah-h^2)-a^2y^2]}$$

$$-\frac{2a}{b}(b^2-y^2)\sin^{-1}\Big(\frac{b(a-h)}{a\sqrt{(b^2-y^2)}}\Big)\Big]dy.$$

Integrating, substituting value of b, and reducing,

$$V_2=\frac{(d-r)[3a^2r^2-(d-r)^2(2ah-h^2)]}{3r\sqrt{(2ah-h^2)}}$$

$$\times\left[\pi-2\sin^{-1}\left(\frac{r(a-h)}{\sqrt{[a^2r^2-(d-r)^2(2ah-h^2)]}}\right)\right]$$

$$+\frac{\pi r^2h(3a-h)}{3(2a-h)}-\tfrac{4}{3}(d-r(a-h)\sqrt{(2dr-d^2)}$$

$$+\frac{4a^3r^2}{3(2ah-h^2)}\tan^{-1}\left(\frac{(a-h)(d-r)}{a\sqrt{(2rd-d^2)}}\right)-\frac{2r^2(a-h)\ (2a^2+2ah-h^2)}{3(2ah-h^2)}\sin^{-1}\left(\frac{d-r}{r}\right).$$

If $d=2r$, $V_2=\dfrac{2\pi r^2h(3a-h)}{3(2a-h)}$.

If $d=2r$, $a=h$, $V_2=\frac{4}{3}\pi r^2h$. Total volume$=V+V_2$.

Also solved by *G. W. GREENWOOD.*

162. Proposed by J. E. SANDERS, Hackney, O.

Solve the differential equations

$$(a)\ x\frac{dy}{dx}-y=x\sqrt{(x^2+y^2)},\ (b)\ \cos x\frac{dy}{dx}+y=1-\sin x.$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College. Philadelphia, Pa.; E. L. SHERWOOD, Professor of Mathematics, Shady Side Academy. Pittsburg, Pa.; M. E. GRABER. Graduate Student, Heidelberg University, Tiffin. O.; and the PROPOSER.

(a) $xdy-ydx=x[x^2+y^2]dx$. Let $y=vx$. $\therefore dy=vdx+xdv$.

$$\therefore dx=\frac{dv}{\sqrt{[1+v^2]}}.\quad \therefore x+A=\log\{v+\sqrt{[1+v^2]}\}.$$

$$\therefore x+A=\log\left(\frac{y+\sqrt{[x^2+y^2]}}{x}\right).$$

(b) $\cos xdy+ydx=[1-\sin x]dx$. Let $y=v[1-\sin x]$.

$\therefore dy=[1-\sin x]dv-v\cos xdx$. $\therefore dx=\cos xdv-v\sin xdx=d[v\cos x]$.

$$\therefore x+A=v\cos x=\frac{y\cos x}{1-\sin x}.\quad \therefore y\cos x=[x+A][1-\sin x].$$

Solved in the same way by *HOMER R. HIGLEY, LON C. WALKER*, and *J. SCHEFFER.*

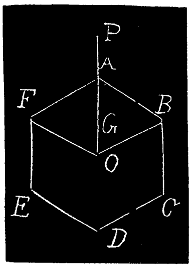
MECHANICS.

153. Proposed by **W. J. GREENSTREET, A. M.**, Editor of The Mathematical Gazette, Stroud, Eng.

An equiangular polygon consisting of equal, freely jointed rods, is hung up from vertex, A. The vertices adjacent to A are connected by a light rod of such length that the polygon is still regular. Find the stress in the rod and the reactions at the vertices.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be the point of suspension, O, the centroid of the system, T, the stress on the weightless rod FB, nW, the weight of the rods, $FB=x$, $AB=a$, $AO=y$, $\angle OAB=\angle OBA=\theta$, R=reaction.



Suppose the deformation such that the centroid moves vertically through a small space. Then $Tdx-nWdy=0$.

$BG:AB=\sin\theta:1$; $\therefore x=2a\sin\theta$.

$AO:AB=\sin\theta:\sin(\pi-2\theta)$; $\therefore y=a\sin\theta/\sin(\pi-2\theta)$.

$\therefore y=a\sin\theta/\sin2\theta=\frac{1}{2}a\sec\theta$.

$\therefore dx=2a\cos\theta d\theta$, $dy=\frac{1}{2}a\sec\theta\tan\theta d\theta$.

$\therefore 2aT\cos\theta d\theta=\frac{1}{2}(nWa)\sec\theta\tan\theta d\theta$.

$\therefore T=\frac{1}{4}nW\sec^2\theta\tan\theta$. R acts along BO, $\angle ABG=\frac{1}{2}\pi-\theta$.

$\therefore \angle OBG=\angle OFG=2\theta-\frac{1}{2}\pi$, $\angle BOF=2\pi-4\theta$.

$\therefore R:T=\sin(2\theta-\frac{1}{2}\pi):\sin(2\pi-4\theta)$; $R:T=\cos2\theta:\sin4\theta$.

$\therefore R=T\cos2\theta/\sin4\theta=T/2\sin2\theta=T/4\sin\theta\cos\theta$. $\therefore R=\frac{1}{16}nW\sec^4\theta$.

Also solved by *G. W. GREENWOOD.*

154. Proposed by **M. E. GRABER**, Graduate Student, Heidelberg University, Tiffin, Ohio.

Find the form of the curve in a vertical plane, such that a heavy bar resting on its concave side and on a peg at a given point, (the origin), may be at rest at all positions.

Solution by **G. W. GREENWOOD, A. B.**, Professor of Mathematics, McKendree College, Lebanon, Ill.; and **E. L. SHERWOOD**, Professor of Mathematics, Shady Side Academy, Pittsburg, Pa.

Let the bar be homogeneous and of length $2b$. Assume the peg and the curve to be smooth. Take the peg as origin, the horizontal line through it in the plane as initial line, and measure θ downwards.

The coördinates of the lower extremity being (r, θ), those of the center of the rod are $(r-b, \theta)$. Since in all positions the rod is in equilibrium, the center is at a constant distance from the initial line. Hence $(r-b)\sin\theta=a$, is the curve traced by one extremity.

Also solved by *G. B. M. ZERR.*

DIOPHANTINE ANALYSIS.

107. Proposed by **L. C. WALKER, A. M.**, Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three positive integral numbers such that the sum of all three of them and the sum of every two of them shall be a square number.

$$4[m_1+m_2+\ldots\ldots+m_{n-1}]^2a_1^2+\{a_1^2-m_1^2+m_2^2+\ldots\ldots+m^2{}_{n-1}-2[m_1m_2+m_1m_3+\ldots\ldots+m_1m_{n-1}]\}^2+\{a_1^2+m_1^2-m_2^2+\ldots\ldots+m^2{}_{n-1}-2[m_1m_2+m_2m_3+\ldots\ldots+m_2m_{n-1}]\}^2+\ldots\ldots+\{a_1^2+m_1^2+m_2^2+\ldots\ldots-m^2{}_{n-1}-2[m_1m_{n-1}+m_2m_{n-1}+\ldots\ldots+m_{n-2}m_{n-1}]\}^2=[n-1][a_1^2+m_1^2+m_2^2+\ldots\ldots+m^2{}_{n-1}]^2\ldots\ldots(8).$$

In (8), let a_1^2, $a_2^2\ldots\ldots a^2{}_n$ represent the first, second.....and nth expressions of the left hand member and a that of the right.

These values of a_1^2, $a_2^2\ldots\ldots a^2{}_n$ in (4), (5).....(6) give the corresponding values of x_1, $x_2\ldots\ldots x_n$.

In the problem, $n=3$; whence

$$a^2=[a_1^2+m_1^2+m_2^2]^2,\ a_1^2=4[m_1+m_2]^2a_1^2\,;$$
$$a_2^2=[a_1^2+m_2^2-m_1^2-2m_1m_2]^2\,;\text{ and}$$
$$a_3^2=[a_1^2+m_1^2-m_2^2-2m_1m_2]^2.$$

These values of a_1^2, a_2^2 and a_3^2 in (4), (5).....and (6) give

$$x_1=4[a_1^2m_2^2+a_1^2m_1m_2-m_1m_2^3+m_1^3m_2]\,;$$
$$x_2=4[a_1^2m_1^2+a_1^2m_1m_2+m_1m_2^3-m_1^3m_2]\,;\text{ and}$$
$$x_3=a_1^4+m_1^4+m_2^4+2m_1^2m_2^2-2a_1^2m_1^2-2a_1^2m_2^2-8a_1^2m_1m_2.$$

Let $a_1=2$, $m_1=3$, and $m_2=4$; then $x_1=112$, $x_2=672$, and $x_3=57$.
$\therefore\ x_1+x_2+x_3=29^2$; $x_1+x_2=28^2$; $x_1+x_3=13^2$; and $x_2+x_3=27^2$.

An excellent solution was also received from *A. H. BELL*.

109. Proposed by HARRY S. VANDIVER, Bala. Pa.

If $m+n+1$ is a prime integer, show that $m!\times n!-(-1)^{\frac{1}{2}(3m-n)}$ is divisible by $m+n+1$. For instance, $6!\times4!-(-1)^7$ is divisible by 11.

Correction and Solution by DR. L. E. DICKSON, The University of Chicago.

In the formula proposed, the exponent of -1 is not correct. Thus, for $m=5$, $n=5$, the number $5!\times5!-(-1)^5$ is not divisible by 11, whereas $5!\times5!-(-1)^4$ is divisible by 11. The correct formula may be stated thus:

$$m!\times(p-1-m)!\equiv(-1)^{m-1}\ (\text{mod}\,p),$$

where p is any odd prime and m any integer $<p$.

The proof follows from the congruences

$$(p-1)!\equiv-1(\text{mod}\,p),\qquad \frac{(p-1)!}{m!(p-1-m)!}\equiv(-1)^m\ (\text{mod}\,p),$$

of which the first is Wilson's Theorem, while the left member of the second is the coefficient of $x^m y^{p-1-m}$ in $(x+y)^{p-1}$. The latter is congruent, modulo p, to

$$\frac{x^p+y^p}{x+y}=x^{p-1}-x^{p-2}y+\dots\dots+(-1)^m x^m y^{p-1-m}+\dots\dots+y^{p-1}.$$

AVERAGE AND PROBABILITY.

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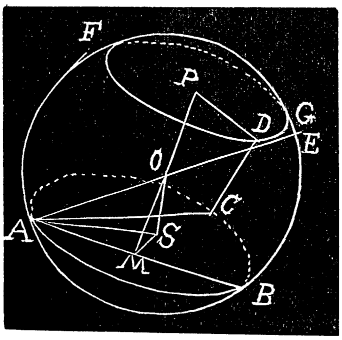
130. Proposed by **LON C. WALKER, A. M.,** Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three and three, is 1-35 of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r; $ABFE$ a great circle through A, B; ABC a small circle through A, B, C, with center S; DGF a small circle through D parallel to ABC, with center P; M the middle point of AB.



Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$AM=r\sin\frac{1}{2}\theta=r_1\cos\psi_1$,

$SM=r\cos\frac{1}{2}\theta\cos\phi=r_1\sin\psi_1$,

$OS=r\cos\frac{1}{2}\theta\sin\phi$,

$AC=2r_1\cos(\psi-\psi_1)$,

$PD=\surd(r^2-x^2)$,

$r_1=r(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{4}\theta\cos^2\phi)^{\frac{1}{2}}$, area $ABC=2rr_1\sin\frac{1}{2}\theta\sin\psi\cos(\psi-\psi_1)$,

volume of tetrahedron D—$ABC=\frac{1}{3}SP.\text{area}ABC=\frac{2}{3}rr_1(x+r\cos\frac{1}{2}\theta\sin\phi)\sin\frac{1}{2}\theta$ $\times\sin\theta\cos(\psi-\psi_1)$.

Hence, we have for the required average volume, $V=\frac{4}{105}\pi r^3=\frac{1}{35}$ of the volume of the given sphere.

[For the integration, see solution in last issue, page 113, where the figure for Professor Zerr's solution was inserted for the one belonging to Professor Walker's solution. F.]

131. Proposed by **LON C. WALKER, A. M.,** Graduate Student, Leland Stanford Jr. University, Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is 10/21 of the volume of the given sphere.

Solution by the PROPOSER.

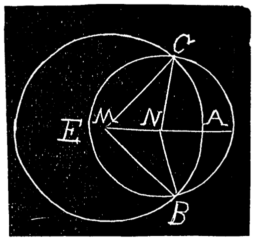
Let M be the center of the given sphere, and N that of the random sphere. Put $BM=a$, $NB=x$, $MN=y$, $\angle MBN=\theta$, $\angle BMN=\phi$, $\angle BNM=\psi$. Then we have

$$\cos\phi=\frac{1}{y}(a-x\cos\theta),$$

$$\cos\psi=\frac{1}{y}(x-a\cos\theta),$$

$$y^2=a^2+x^2-2ax\cos\theta.$$

Volume of spherical section $BACM=\frac{2}{3}\pi a^3(1-\cos\phi)=\frac{2}{3}\pi[a^3-1/y(a^4-a^3\cos\theta)]$;
volume of spherical sector $BECN=\frac{2}{3}\pi x^3(1-\cos\psi)=\frac{2}{3}\pi[x^3-1/y(x^4-ax^3\cos\theta)]$;
volume of solid $BMCN=\frac{1}{3}\pi a^2y\sin^2\phi=\frac{1}{3}\pi(a^2x^2/y)\sin^2\theta$;
volume of solid $BACE=S=\frac{1}{3}\pi[2a^3+2x^3-2/y\,(a^4+x^4)+2/y\,(a^3x+ax^3)\cos\theta-(a^2x^2/y)\sin^2\theta]$.



Hence we have for the required average volume,

$$V=\frac{\int_0^a\int_{a-x}^a S.4\pi y^2dxdy+\int_a^{2a}\int_{x-a}^a S.4\pi y^2dxdy}{\int_0^a\int_{a-x}^a 4\pi y^2dxdy+\int_a^{2a}\int_{x-a}^a 4\pi y^2dxdy}$$

$$=\frac{2\pi}{3a^4}\Big\{\int_0^a\int_{a-x}^a\Big[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta-a^2x^2y\sin^2\theta\Big]dxdy$$

$$+\int_0^{2a}\int_{x-a}^a\Big[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta-a^2x^2y\sin^2\theta\Big]dx\,dy\Big\}$$

$$=\frac{2\pi}{3a^4}\Big\{\int_0^{2a}\Big[\tfrac{2}{3}a^6-2a^5x+2a^4x^2+\tfrac{5}{4}a^2x^4-2ax^5+\tfrac{17}{24}x^6\Big]dx$$

$$+\int_0^a\tfrac{2}{3}(a^3+x^3)(x-a)^3dx+\int_a^{2a}\tfrac{2}{3}(a^3+x^3)(a-x)^3dx\Big\}$$

$=\frac{40}{63}\pi a^3=\frac{10}{21}$ of the volume of the given sphere.

Also solved by *G. B. M. ZERR*.

132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1-(1/2^{n-2})$.

Solution by the PROPOSER.

Using a figure similar to the one used in problem 116, let $AOD=\theta_1$, $BOD=\theta_2$, $COD=\theta_3$, etc. Let p be the required chance, p_1, the chance that it does not contain the center of the circle.

$$p_1=\frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\theta_1}\int_0^{\theta_2}\ldots\int_0^{\theta_{n-2}}d\theta_1 d\theta_2 d\theta_3\ldots d\theta_{n-1}}{\int_0^{\pi}\int_0^{\theta_1}\int_0^{\theta_2}\ldots\int_0^{\theta_{n-2}}d\theta_1 d\theta_2 d\theta_3\ldots d\theta_{n-1}}=\frac{(n-1)!}{\pi^{n-1}}.\frac{2(\frac{1}{2}\pi)^{n-1}}{(n-1)!}=\frac{1}{2^{n-2}}.$$

$$p=1-p_1=1-(1/2^{n-2}).$$

Also solved with the same result by *J. SCHEFFER.*

MISCELLANEOUS.

133. Proposed by HARRY S. VANDIVER, Bala. Pa.

If a group G of order mn has a subgroup H of order n, and if n has no prime factor which is less than m, show that H must be a self-conjugate subgroup. (Frobenius.)

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia. Pa.

Let the substitution of the given subgroup (H) be 1, t_2, t_3, ...t_p.

If H is not self-conjugate, multiply it into any substition (S) which is not commutative to it. If the transform (H_1) of (H) with respect to any one of these products contains a substitution ($s^{-1}ts$) of order m^β ($\beta>0$) which is not found in H and if t does not transform H_1 into itself we can form the following rectangle with m conjugate rows of p elements:

$$\begin{array}{l} s,\ st_2,\ st_3,\ \ldots st_p \\ ts,\ tst_2,\ tst_3,\ \ldots tst_p \\ st^2,\ t_2st^2,\ t_3st^2,\ \ldots t_pst^2 \\ \ldots\quad \ldots\quad \ldots \\ st^{m-1},\ t_2st^{m-1},\ t_3st^{m-1},\ \ldots t_pst^{m-1}. \end{array}$$

All the substitutions of a given row transform H into the same group, while any two substitutions from different rows transform H into two different groups. As no substitutions in the given rectangle can be equal to each other, the entire group would have to contain at least $p(m+1)$ substitutions. This is contrary to the hypothesis. Therefore H is a self-conjugate sub-group.

134. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Give a complete solution of the Jacobian equation $\kappa^2\mathrm{sn}^4u+2\kappa^2\mathrm{sn}^2u+1=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let sn$u$=x. $\therefore$ $x^4+2x^2+1/\kappa^2=0$, where κ is the modulus.

$\therefore$ $(x^2+1)^2=-(1-\kappa^2)/\kappa^2$.

$\therefore$ $x=\pm\sqrt{\dfrac{-\kappa\pm\sqrt{(-1)}\sqrt{(1-\kappa^2)}}{\kappa}}=\pm n$, suppose. $\therefore$ sn$u=\pm n$.

$$\therefore u=2\int_0^n \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}=2F(\kappa,\, n)$$

$$=2[n+\tfrac{1}{6}n^3(1+\kappa^2)+\tfrac{1}{40}n^5(3+2\kappa^2+3\kappa^4)+\ldots\ldots].$$

When $\kappa=1$, $u=2\sqrt{-1}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots)$, since $n=\pm\sqrt{-1}$).

$\therefore$ $u=\frac{1}{2}\pi\sqrt{-1}$.

But $u=2\displaystyle\int_0^{\sqrt{-1}}\frac{dx}{1-x^2}=\log\left(\frac{1+\sqrt{-1}}{1-\sqrt{-1}}\right)$.

$\therefore$ $\pi=2\sqrt{-1}\log\left(\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}\right)$, a result previously referred to in this journal.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

167. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A traveler notices that $m=2\frac{1}{2}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of train in miles per hour. How far apart are the poles?

168. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I buy for $m=10\%$ and $n=5\%$ *less* I shall gain $p=15\%$ and $q=5\%$ *more*. What is my rate of gain?

ALGEBRA.

178. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A_n being the arithmetic mean of the nth powers of the numbers less than p and prime to it, find a relation between A_3, A_2 and p.

179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left(\frac{q^2}{p}-\frac{p^3}{5q}\right)=0.$$

180. Proposed by the late JOSIAH H. DRUMMOND.

If r/s is such a value of p as makes $m/(p^2-2)$ integral, prove that $(3r+4s)/(2r+3s)$ is another such value, so that an indefinite number of integral values may be obtained.

Also, if r/s is such a value of p as makes $2m/(p^2-2)$ integral, prove that $2(r+s)/(r+2s)$ is also such a value.

GEOMETRY.

201. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Two plane sections of a right circular cone have their major axes AA', aa' coplaner, and Aa on one generator equal to $A'a'$ on the other. The projections of the sections on any plane perpendicular to the axis are confocal.

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The equations $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$ and $l\beta\gamma+m\alpha\gamma+n\beta\alpha=0$ represent ellipses. If a, b, c are the sides of the triangle of reference, transform to Cartesian coördinates and find area of each ellipse.

CALCULUS.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $(y^2+x^2)=a^2(x^2-y^2)$ round the axis of x.

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Integrate, $$\int_0^a \int_0^b \int_0^c \frac{z\,dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

MECHANICS.

157. Proposed by T. W. WRIGHT, Schenectady, N. Y.

Explain why a waterfall h feet high can support a column of water $2h$ feet high.

158. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are electrical images of each other in respect to a given sphere may be constant, and that their centers of gravity shall lie on the surface of the sphere.

AVERAGE AND PROBABILITY.

143. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

The extremities of two equal lines drawn from a fixed point in the circumference of a given circle is joined. Find the average area of the circle inscribed in the triangle formed.

144. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

In a circular park 400 feet in diameter are 4 *equal* circular ponds of *variable* diameter. What is the probability that a sightless person walking in a straight line from the center of the park, will step into a pond ?

MISCELLANEOUS.

138. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Find an invarient of the *third degree* in the coefficients of a ternary quartic.

139. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Given the roots of a binary cubic, to find the roots of its two independent covariants.

NOTES.

Professor W. F. Osgood of Harvard University, has been promoted to a full professorship of mathematics. F.

Dr. C. A. Noble has been promoted to an assistant professorship of mathematics at the University of California. F.

Professor Alexander Macfarlane delivered, at Lehigh University, April 20-23, a course of six lectures on the British mathematicians, Kirkham, Babbage, Whewell, Dodgson, Stokes, and Rayleigh. F.

Professor John J. Quinn has brought to public attention a third triangle, to be used with the two triangles commonly used in drawing sets, and in a small circular illustrates many constructions which are easily made by means of this triangle of which he is the inventor. F.

Professor Josiah Willard Gibbs, of Yale University, died at New Haven, April 28th, 1903, of heart disease. Professor Gibbs was born in New Haven, Feb. 11, 1839, and graduated at Yale in 1858. In 1863, he received the degree of Doctor of Philosophy. After studying in Paris, Berlin, and Heidelberg, he was appointed, in 1871, to the Professorship of Mathematical Physics in Yale, which position he held until the time of his death. He was a member of the Royal Society of London, of the National Academy of Science, of the American Mathematical Society, and many other learned bodies. He was an authority of the first rank in thermo-dynamics, and in the application of vector analysis to physical problems. Last year, 1902, he published a work entitled *Elementary Principles in Statistical Mechanics.* F.

BOOKS AND PERIODICALS.

Traité de Géométrie. Per C. Guichard, Professeur L'Université de Clermont, Deuxième Partie, Compléments. Paper bound. Price, 6 fr. Paris: Librarie Nony & Cie.

This is Volume II of the Treatise on Geometry. The first volume treats of plane and solid geometry. The second volume is divided into five sections. The first section is divided into nine chapters, and some of the subjects here treated in a very excellent way are, Geometry of Systems of Lines, Systems of Circles, Transversals, Reciprocal Polars, Involution, Inversion, Tangent Circles, and the Nine Point Circle. The second section contains, among other subjects, Orthogonal Projections, Theory of Vectors, and Central Projections. The third part treats of Poles and Polar Planes with Respect to a Sphere, Inverse Figures in Space, Orthogonal Spheres, and Geometry of the Sphere. The fourth part treats of the Conic Sections, and the fifth part of Geometry of Position and the Measurement of Areas. The work contains many interesting theorems and discussions not found in the ordinary text-book on Geometry. F.

Quinn's Geometry Tablet, Adapted to Geometry and Physics. Scranton, Wetmore & Co., Rochester, N. Y.

This Tablet is adapted to formal written work in original demonstrations; a few sheets at the end of the tablet are intended for construction problems. By its use the labor of examining students' daily written work should be materially reduced.

Annals of Mathemathics. Published quarterly, and under the auspices of Harvard University. Price, $2.00 per year, in advance.

The April number contains the following articles: The Cardiod and Tricuspid; Quartics with Three Cusps, by R. C. Archibald; Note on a Partial Differential Equation of the First Order, by Dr. E. D. Roe, Jr.; On a Generalization of the Set of Associated Minimum Surfaces, by A. S. Gale; Twisted Quartic Curves of the First Species, and Certain Covariant Quartics, by H. S. White; On the Characteristics of Differential Equations, Part I, by E. R. Hendrick.

The American Journal of Mathematics. Published, quarterly, under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance.

Numbsr 2, Volume XXV, contains the following articles: The Double Six Configuration Connected with the Cubic Surface, and a Related Group of Cremona Transformations, by Edward Kasner; Untersuchungen über Linear Differentialgleichungen 4 Ordnung und die zugehhörigen Gruppen, Von Saul Epsteen; The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers, by A. W. Whitehead; On Differential Equations Belonging to Ternary Linearoid Groups, by F. E. Ross; On a Certain Group of Isomorphisms, by J. W. Young.

The Mathematical Gazette. Edited by J. W. Greenstreet, Stroud, England.

The May number contains several important articles, among which are On the Representation of Imaginary Points by Real Points in the Plane, and An Elementary Introduction to the Infinitesimal Calculus of Surfaces.

THE AMERICAN MATHEMATICAL MONTHLY.

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THE APOLLONIAN PROBLEM IN SPACE.

By EDWARD KASNER.

The so-called Apollonian problem, namely, the construction of a circle tangent to three given circles in a plane, is among the most famous in the domain of elementary geometry. It was proposed (and probably solved) by Apollonius of Perga, in one of his last works, *On Contact;* but the first solution that has come down to us is that given by Vieta in his attempted restoration which he entitled *Gallus Apollonius* (1600).* Since then it has been treated in an almost endless variety of ways, for instance, by Gergonne, Pluecker, Steiner, Hart, Casey, and recently in an elaborate memoir by Study (*Mathematische Annalen*, vol. 49).

The analagous problem in solid geometry is usually taken to be the construction of a sphere tangent to four given spheres.† There is, however, a different extension, which is the subject of this note, namely, the construction of a circle tangent to two given circles in general position in space. That this is a definite problem is seen most simply by an enumeration of constants. A circle in space requires six parameters for its determination, for example, three to fix its plane, two to fix its center in the plane, and one to fix its radius. The number of circles tangent to a given circle is ∞^3; for there are ∞^2 points on the circle, at each of which there are ∞^2 tangent planes, and in each of these planes there are ∞^1 tangent circles. Hence, tangency of circles in space is a three-fold con-

*In response to Vieta's proposal of the problem, a solution was given by Romanus which is however unsatisfactory, as it makes use of conic sections instead of the straight edge and compass only. See Cantor, *Geshichte der Mathematik*, 2nd edition, Vol. II, page 590.

†This was first considered by Fermat in his *De contactibus sphaericis*. See Cantor, Vol. II, page 659.

dition, instead of a simple condition as it is for circles in a plane or spheres in space. If a circle is to be tangent to two given circles, this is equivalent to the imposition of six simple conditions, so that the number of solutions is finite. It will be seen, in fact, that there are four solutions.

Denote the two given circles by C' and C'', their planes by π' and π'', and the line in which the latter intersect by l. The first step in the solution is to construct the sphere S which cuts both C' and C'' othogonally. The center of S must obviously lie in both π' and π'', and hence in l. Moreover, it must be so situated that the tangents from it to C' are equal to those drawn to C''. Hence the center may be obtained as follows: Revolve the circle C'' about the line l as axis until it falls in the plane π'; denote the circles so obtained by F; construct the radical axis of C' and F; this will intersect l in the point P required. For the radical axis is the locus of points for which the tangents drawn to the circles C' and F are equal; and every point in l has the property that the tangents drawn from it to C'' and F are equal. The sphere S has its center at P, and its radius is equal to either of the tangents from P to C' or C''. This construction also shows that S is unique.

Denote the two points in which C' intersects S by A', B'; and similarly the points in which C'' intersects S by A'', B''. Select one point from the first pair, and one from the second pair. Through the two points selected there passes a unique circle* orthogonal to S. This circle will be tangent to both C' and C''. In this way we obtain four circles touching both of the given circles, corresponding to the four possible pairs of points, A', A''; A', B''; B', A''; B' B''.

It remains to be shown that there are no additional solutions of the problem. Let C be any circle tangent to C' and C'' at, say, the points P' and P'' respectively. Construct the tangent lines to C at P' and P'', and denote their point of intersection by Q. With Q as center and radius $QP'=QP''$ describe a sphere. This sphere will be orthogonal to C at P' and P'', hence it is also orthogonal to C' and C''. From what precedes it follows that the sphere constructed coincides with the orthogonal sphere S. Therefore P' must lie at either A' or B', and P'' must lie at either A'' or B'', so that C necessarily coincides with one of the four circles constructed above.

The two given circles and the four circles tangent to them form an interesting set of six circles. To bring out the symmetry of the configuration, it will be convenient to introduce a new notation. Let the four points of intersection on the sphere S be denoted by P_1, P_2, P_3, P_4. Each of the six circles intersects S in two of these points, and conversely to any pair of points there corresponds one of the circles. Denote the circle corresponging to P_i, P_k by C_{ik}. The circles fall naturally into three pairs:

(I)	C_{12}, C_{34};
(II)	C_{13}, C_{24};
(III)	C_{14}, C_{23}.

*To obtain this circle, construct the great circle on S determined by the two points; the required circle then lies in the plane of this great circle and is orthogonal to it at the same two points.

The two circles of any of these pairs are in general position; the four circles tangent to them are those belonging to the other two pairs. Hence any one of the six circles is touched by four of the remaining circles, namely, by all except the one belonging to the same pair.

Pass a circle through any three of the four points of intersection, say P_2, P_3, P_4. This circle determines a unique sphere, passing through it and orthogonal so S, which may be denoted by S_1. This sphere, it is easily shown, passes through the circles C_{23}, C_{24}, C_{34}. In all there are four spheres, S_1, S_2, S_3, S_4 of this kind, each containing three of the six circles.

Let us consider the different ways in which a triple of circles can be selected.

1°. Two circles of the same pair and any other circle, for example, C_{12}, C_{34}, C_{13}; two of the circles are then in general position and the third touches both.

2°. One from each of the three pairs, but in such a way that the three circles have no common point, for example, C_{12}, C_{13}, C_{23}; the circles are then co-spherical (lying on one of the spheres S_1, S_2, S_3, S_4) and are mutually tangent.

3°. Three having a point in common, for example, C_{12}, C_{13}, C_{14}; the circles are then tangent to one another at the common point.

The numbers of triples for the three types are respectively 12, 4 and 4. If we consider in each case the residual triple, for 1° it is of the same type; while for 2° it is of type 3° and *vice versa.*

The problem considered belongs essentially to the geometry based upon the group of all conformal transformations in space, namely, the group generated by inversions or transformations by reciprocal radii vectors. From the point of view of this geometry the infinite region of space is regarded as a point, instead of a plane, as in projective geometry; a straight line is thought of as a special circle, namely, one which passes through the infinite point; and similarly a plane is a special sphere.

In the preceding discussion of the problem, it was assumed that the two given circles were in general position. The enumeration of all the special cases is not difficult and makes an interesting exercise, It will suffice to state that, if the two circles intersect at a single point (including, for instance, the case where the circles become two straight lines in general position), there is no proper solution (all the four circles shrinking up into the point of intersection). On the other hand, if the circles have two real or imaginary points in common, *i. e.*, if they are co-spherical, the problem becomes indeterminate, for then there are an infinite number of tangent circles just as in the case of two circles in plane geometry.

Columbia University, *May 7, 1903.*

ON THE DEFINITION OF AN INFINITE NUMBER.

By DR. G. A. MILLER.

Mathematics is replete with surprises. Three of these are especially noteworthy on account of the fundamental principles involved, namely, (1) There are perfectly consistent geometries in which the sum of the angles of a plane triangle is not equal to two right angles. (2) There are algebras in which the commutative law of multiplication does not hold. (3) There are aggregates such that a part is equivalent to the whole.

The first of these has been frequently noted in this journal. In regard to the second, Poincaré recently said*: "Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative; *i. e.*, the product is changed when the order of the factors is changed. This presents a revolution in arithmetic which is entirely similar to the one which Lobatchevski effected in Geometry."

The object of the present note is to give some details in regard to an infinite aggregate. The method employed to compare two aggregates is one of the most rudimentary ones. In fact, it is essentially the same as that employed in the earliest stages of civilization when objects are counted by means of pebbles. In this way a (1, 1) correspondence is established between a given pile of pebbles and a certain aggregate of objects.

Similarly, two aggregates or totalities are said to be *equivalent* when it is possible to establish a (1, 1) correspondence between the units of the aggregates. That is, when the units of the aggregates can be so associated that to each unit of one of the aggregates there corresponds one and only one unit of the other and *vice versa.* For instance, every even integer is of the form $2n$, and every odd integer is of the form $2n+1$. For any integral value of n we obtain one and only one even number, and also one and only one odd number. By associating these two numbers for every value of n we obtain a (1, 1) correspondence between the odd and the even numbers. Hence we say that the totality of even numbers is equivalent to the totality of odd numbers.

A more interesting illustration may be given by means of the equation $y=1/(x+1)$. Suppose we assign to x any positive rational value. The corresponding value of y will always be a positive rational number which does not exceed unity. Whenever we give x two distinct values, y will assume two distinct values in the given intervals and *vice versa.* Hence it follows that *the rational positive numbers which do not exceed unity are equivalent to all the rational positive numbers.*

By employing the equation $2y=1/(x+1)$, it is easy to see that the totality of the positive rational numbers which do not exceed one-half is equivalent to

*Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 250.

the totality of all the positive rational numbers. More generally, the equation $ay=1/(x+1)$, a being a positive rational number, indicates that the totality of the positive rational numbers which do not exceed an arbitrary finite limit is equivalent to the totality of all the positive rational numbers.

The above examples furnish some of the most important instances of multitudes in which a part is equivalent to the whole. It should be observed that this equivalence is proved by one of the most elementary methods of comparison—a method which every one employs with perfect confidence.

With respect to any aggregate only two cases are thinkable; either a part is equivalent to the whole or there is no such part. In the former case the aggregate is said to be infinite, in the latter case it is said to be finite. This definition of an infinite number of things is due to Dedekind and is of fundamental importance.* That infinite numbers have this property was noted by others, but Dedekind was the first to use it as a definition.

In employing this definition any aggregate (M_1) is to be considered a part of another aggregate (M_2), provided each of the units of M_1 is contained in M_2, while there is at least one unit in M_2 which is not contained in M_1. For instance, all the even integers are a part of the total number of integers and all the integers are a part of all the rational numbers.

It is now easy to prove that the number of integers is infinite, for the equation $y=2x$ shows that there is just one even integer (y) which corresponds to any integer (x), and that there is just one integer x which corresponds to any even number y. That is, there is a (1, 1) correspondence between the total number of integers and the even integers. As the latter constitute a part of the former it follows that a part of the totality of integers is equivalent to the whole number of integers, that is, the number of integers is infinite.†

By the same method it may be proved that time, according to the common conception, is infinite; for there is a (1, 1) correspondence between the total number of hours and the total number of half hours. If the half hours were denoted by the natural numbers the hours would correspond to the even numbers. Similarly, it may be observed that space (defined analytically by coördinates) is infinite according to the given definition.

If it is possible to establish a (1, 1) correspondence between two infinite totalities they are said to be of the same power. The totality (M_0) formed by the natural numbers is especially important. Any totality which has the same power as M_0 is said to be numerable (countable). It is an extremely interesting fact that all algebraic numbers are countable. In particular, the natural numbers are equivalent to all the rational numbers. These facts constitute some of the elements of the important subject known as the theory of aggregates (ensemble, Mengenlehre) to which the reader may be referred for further developments.

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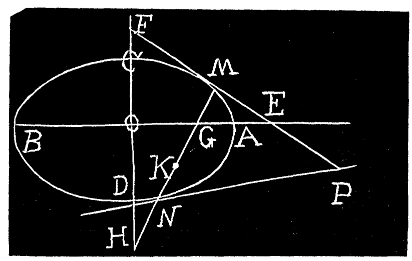
*Dedekind, *Essays on Number*, translated by Beman, 1901, page 63.

†This result can be proved in an infinite number of ways by using the equation $y=ax$, a being any integer. It may also be proved by the equations $y=x^a$, $y=x+a$, etc.

CERTAIN LOCI RELATED TO A CONIC.

By G. B. M. ZERR.

Let M be any point on an ellipse, axes $2a$, $2b$, eccentricity e. O the center of the ellipse, MP the tangent, MN the normal, and MQ the diameter, respectively, at the point M. Let E, F be the points where the tangent MP intersects the axes; G, H the points where the normal intersects the axes. P the pole of MN; K the center of curvature. Let (u, v) be the coördinates of M. Then $(-u, -v)$; $(a^2/u, 0)$; $(0, b^2/v)$; $(e^2u, 0)$; $(0, -a^2e^2v/b^2)$; $(e^2u^3/a^2, -a^2e^2v^3/b^4)$; $[a^2/(e^2u), -b^4/(a^2e^2v)]$; $[u(1-b^2p), v(1-a^2p)]$, where $p=2\left(\frac{a^4v^2+b^4u^2}{a^6v^2+b^6u^2}\right)$, are the coördinates of Q, E, F, G, H, K, P, and N, respectively.



The equation of the perpendicular from O on MP is $y=a^2vx/b^2u$.....(1).

The perpendicular from K on OG is $x=e^2u^2/a^2$.....(2).

The perpendicular from G on OM is $y+\frac{u}{v}(x-e^2u)=0$.....(3).

The perpendicular from K on OH is $y+a^2e^2v^3/b^4=0$.....(4).

The perpendicular from H on OM is $y+a^2e^2v/b^2+ux/v=0$.....(5).

The perpendicular from E on MP is $y=\frac{a^2v}{b^2u}\left(x-\frac{a^2}{u}\right)$.....(6).

The perpendicular from K on MN is $y+\frac{a^2e^2v^3}{b^4}+\frac{b^2u}{a^2v}\left(x-\frac{e^2u^3}{a^2}\right)=0$.....(7)

The perpendicular from F on MP is $y=b^2/v+a^2vx/b^2u$.....(8).

The perpendicular from K on OM is $y+\frac{a^2e^2v^3}{b^4}+\frac{u}{v}\left(x-\frac{e^2u^3}{a^2}\right)=0$..... (9).

The perpendicular from H on NM is $y+\frac{a^2e^2v}{b^2}+\frac{b^2ux}{a^2v}=0$.....(10).

The perpendicular from G on MN is $y+\frac{b^2u}{a^2v}(x-e^2u)=0$.....(11).

The perpendicular from M on OG is $x=u$.....(12).

The perpendicular from M on OH is $y=v$.....(13).

The perpendicular from M on OM is $y-v+\frac{u}{v}(x-u)=0$.....(14).

The line PG is $y(a^4v-a^2e^4u^2v)+b^4ux-b^4e^2u^2=0$.....(15).

The line PH is $y+\frac{a^2e^2v}{b^2}=\frac{(a^4e^4uv^2-b^6u)x}{a^4b^2v}$.....(16).

The line QN is $a^4vy+b^4ux+a^4v^2+b^4u^2=0$.....(17).

The line OK is $y+\frac{a^4v^3x}{b^4u^3}=0$.....(18).

The line through the projections of Q on OG and OH is $uy+vx+uv=0$.....(19).

Then we have the following eleven sets of three concurrent lines:
1. (1), (2), (3) meet in $x=e^2u^3/a^2$, $y=u^2v/b^2$.

$\therefore\ u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=b^2y\sqrt[3]{\frac{e^4}{a^4x^2}}$. Substituting these values of u, v in $b^2u^2+a^2v^2=a^2b^2$, we get for the locus of this point $(a^2x^2+b^2e^4y^2)^3=a^8e^4x^4$.

2. (1), (4), (5) meet in $x=-\frac{e^2uv^2}{b^2}$, $y=-\frac{a^2e^2v^3}{b^4}$.

$\therefore\ u=-\frac{ax}{e}\sqrt[3]{\frac{ae}{b^2y^2}}$, $v=-b\sqrt[3]{\frac{by}{a^2e^2}}$.

$\therefore$ The locus of this point is $(a^2x^2+b^2y^2)^3=a^4b^4e^4y^4$.

3. (9), (10), (12) meet in $x=u$, $y=-\frac{a^4e^2+b^2u^2-a^2e^2u^2}{a^2v}$.

$\therefore u=x, v=-\frac{a^2e^2+b^2x^2-a^2e^2x^2}{a^2y}$.

The locus of this point is $a^2b^2x^2y^2+(a^4e^2+b^2x^2-a^2e^2x^2)^2=a^4b^2y^2$.

4. (9), (11), (13) meet in $x=\frac{a^2b^2e^2-a^2e^2v^2-a^2v^2}{b^2u}$, $y=v$.

$\therefore\ u=\frac{a^2b^2e^2-a^2e^2y^2-a^2y^2}{b^2x}$, $v=y$.

The locus of this point is $a^2(b^2e^2-e^2y^2-y^2)^2+b^2x^2y^2=b^4x^2$.

5. (2), (8), (14) meet in $x=\frac{e^2u^3}{a^2}$, $y=\frac{a^2b^2+a^2e^2u^2-e^2u^4}{a^2v}$.

$\therefore\ u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=\left(b^2+ae\sqrt[3]{\frac{ax^2}{e}}-x\sqrt[3]{\frac{a^2x}{e^2}}\right)/y$.

Let $x=z^3$, $a/e=c^3$, then the locus of this point is $b^2c^4y^2z^2+a^2(b^2+acez^2-c^2z^4)^2=a^2b^2y^2$.

6. (4), (6), (14) meet in $x=\dfrac{a^2b^4-a^2b^2e^2v^2+a^2e^2v^4}{b^4u}$, $y=-\dfrac{a^2e^2v^3}{b^4}$.

$\therefore\ u=\left(a^2-ae\sqrt[3]{\dfrac{y^2}{ae}}+y\sqrt[3]{\dfrac{y}{a^2e^2}}\right)/x$, $v=-b\sqrt[3]{\dfrac{y}{a^2e^2}}$.

Let $y=z^3$, $1/ae=c^3$. $\therefore\ u=(a^2-aecz^2+c^2z^4)/x$, $v=-bc^2z$.
The locus of this point is $(a^2-acez^2+c^2z^4)^2+a^2c^4x^2z^2=a^2x^2$.
The loci of 5 and 6 can be found in terms of x and y by expansion.

7. (7), (12), (16) meet in $x=u$, $y=\dfrac{a^2e^2(a^2-u^2)(e^2u^2-a^2)-b^4u^2}{a^4v}$.

$\therefore\ u=x$, $v=(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)/a^4y$.
The locus of this point is $a^6b^2x^2y^2+(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)^2=a^8b^2y^2$.

8. (7), (13), (15) meet in $x=\dfrac{a^2(a^2b^2e^4v^2-a^2e^4v^4-a^2b^2v^2+b^6e^2-b^4e^2v^2)}{b^6u}$

$y=v$. $\therefore\ u=\dfrac{a^2(a^2b^2e^4y^2-a^2e^4y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)}{b^6x}$, $v=y$.

The locus of this point is $a^2(a^2b^2e^4y^2-a^2e^2y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)^2+b^{10}x^2y^2=b^{12}x^2$.

9. (3), (6), (7) meet in $x=\dfrac{b^2e^2u^4+a^4v^2}{a^2b^2u}$, $y=-\dfrac{b^4v+a^2e^2v^3}{b^4}$.

Also, $x=\dfrac{a^2b^4e^2-2a^2b^2e^2v^2+a^2e^2v^4+a^2b^2v^2}{b^4u}$.

10. (5), (7), (8) meet in $x=-\dfrac{a^2u-e^2u^3}{a^2}$, $y=\dfrac{b^6u^2-a^4e^2v^4}{a^2b^4v}$.

Also, $y=\dfrac{b^2u^2-a^4e^2+2a^2e^2u^2-e^2u^4}{a^2v}$.

11. (17), (18), (19) meet in $x=\dfrac{b^4u^3}{a^4v^2-b^4u^2}$, $y=-\dfrac{a^4v^3}{a^4v^2-b^4u^2}$.

Also, $x=\dfrac{b^2u^3}{a^4-(a^2+b^2)u^2}$, $y=\dfrac{a^2v^3}{b^4-(a^2+b^2)v^2}$.

The loci of 9, 10, 11 can be found by solving cubic equations or by means of Calculus. In case 11, it may also be solved as follows:

$$x/y=-\frac{b^4u^3}{a^4v^3} \text{ or } u/v=-\frac{a}{b}\sqrt[3]{\frac{ax}{by}}. \quad \therefore\ u=-\frac{av}{b}\sqrt[3]{(ax/by)}, \quad v=-\frac{bu}{a}\sqrt[3]{(by/ax)},$$

from values of x, y we get $u+x=\frac{a^2x}{b^2}\sqrt[3]{\frac{b^2y^2}{a^2x^2}}$, $v+y=\frac{b^2y}{a^2}\sqrt[3]{\frac{a^2x^2}{b^2y^2}}$.

These relations also apply to the hyperbola.

ON CERTAIN PROOFS OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

By DR. ROBERT E. MORITZ, Assistant Professor in the University of Nebraska.

I.

Until an American text-book on Higher Algebra shall appear, the great majority of American students will probably continue to approach the Fundamental Theorem of Algebra through the well known English texts of Chrystal, Burnside and Panton, or Todhunter. It is therefore to be regretted that these texts, though they aim at considerable rigor in demonstration, fail when it comes to this most important theorem. Yet it is this very theorem where rigor means all, for the mere fact which the theorem embodies, is well known to every student long before he reaches the demonstration in question. It is my pnrpose to point out as briefly as possible certain of these tacit assumptions employed by the several authors, in the hope that if it is not desirable to entirely avoid them, as has been done by Weber in his classic text on algebra, they may at least be explicitly stated as such, in future editions of these and other texts.

II.

Chrystal's* proof is in outline as follows: To prove that one value of z, in general a complex number, can always be found which causes the rational integral function, $f(z)$, to vanish, "we have to show that a value of z can always be found which shall render mod $f(z)$ smaller than any assignable quantity. This will be established if we can show that however small mod $f(z)$ be, provided it be not zero, we can always, by properly altering z, make mod $f(z)$ smaller still." The proof now consists in showing that so long as $f(z)\neq 0$, an increment h of z may be so determined that

$$\text{mod } f(z+h) < \text{mod } f(z);$$

*Chrystal, *Algebra*, I, Chapter XII, §22.

that is, so long as mod$f(z)$ is not zero, it may be diminished, and hence it is concluded that one value of z can be found such that mod$f(z)=0$, that is, such that $f(z)=0$.

The assumption here is, that if for every value of $f(z)$ an h can be found such that mod $f(z+h)<$mod $f(z)$, $f(z)$ must necessarily vanish for some value of z. This is a non sequitur. The inference is not warranted that a function which permits of diminution for every value of the argument possesses necessarily a zero value. If, for example,

$$f(x)=\left(\frac{1+x^2}{x^2}\right)^{x^2+1},$$

$f(x)$ has no zero value, yet for every value of x an h may be found such that

$$f(x+h)<f(x).$$

III.

Burnside and Panton* retain through successive editions of their work the following proof: Suppose $z=x+iy$ is represented by the point (x, y) in the z-plane and its image $w=f(x+iy)=u+iv$ by the point (u, v) in the w-plane. If a point $z_1=z+h$, sufficiently near z, is made to describe a small closed curve about z, its image $w_1=w+k$ will describe a corresponding small closed curve about w. Suppose, now, that there is no value of z which makes $f(z)=0$, then there must be some point z, whose image w is nearer the origin than that of any other point z. But by properly selecting the path of z_1, the path of its image w_1 can be made to pass between w and the origin. This is contrary to the supposition that w is the nearest possible position with reference to the origin, consequently no value different from zero can be the least possible value of mod $f(z)$.

The unwarranted assumption in this proof is, that there must be some z whose image w is necessarily nearer the origin than that of any other z. Nor is the objection removed by letting w be the image nearer the origin (or at least as near as) the image of any other point z, as has been done by Fine† in his otherwise admirable proof of this theorem. The real objection is that w is assumed to possess a nearest position to 0 at all, that is, that the function mod $f(z)$ is assumed to possess a minimum, a supposition the contrary of which is certainly conceivable and may be legitimately held so long as nothing to the contrary has been established. In fact, it is possible to construct an algebra in which the assumption does not hold. We need only to confine ourselves to the domain of real numbers which are greater than unity. Let x be any number of this domain, then

$$f(x)=\frac{x^3+2x}{1+2x^2}$$

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*Burnside and Panton, *Theory of Equations*, Fourth Edition, Vol. I, §123.
†Fine, *The Number-system of Algebra*, §54.

has no minimum value in this domain. For so long as x is real and less than unity, $f(x)$ also is real and less than unity, and

$$f(x) < x,$$

that is, for every conceivable value of $f(x)$ of our domain we can construct another which is less.

IV.

Todhunter,* in spite of his extreme caution in dealing with demonstrations of a critical nature, also assumes the existence of a minimum without recognizing it. He himself outlines his argument in the words, "Since U^2+V^2 (that is, mod $f(z)$) is always a real positive quantity, if it can not be zero there must be some value which can not be diminished; but we shall now prove that if U^2+V^2 have any value different from zero we can diminish that value by a suitable change in the expression which is substituted for x, so that it follows that U^2+V^2 must be capable of the value zero, that is, U and V must vanish simultaneously." The argument as here stated really involves two unwarranted assumptions:

1. That the function U^2+V^2 necessarily possesses a minimum,
2. That a function which permits of being indefinitely diminished must necessarily approach the limit zero.

The second of these assumptions is removed in a postscript article, but the first is ignored.

It may be mentioned in conclusion that the assumption of the existence of a minimum occurs in the common source of the various proofs which I have cited, the so-called first proof of Cauchy,† or going back still further in Legendre‡ who first developed the essential principles of Cauchy's proof.

THE UNIVERSITY OF NEBRASKA, *May, 1903.*

A GENERAL NOTATION FOR VECTOR ANALYSIS.

By JOSEPH V. COLLINS, Stevens Point, Wis.

Vector analysis is now a little over half a century old. As compared with most other branches of mathematical thought its cultivation is very recent. As we look back over the history of the development of the mathematical notations we see that at first there was great diversity in the signs used, but that this diversity in time gave place to uniformity.

*Todhunter, *Theory of Equations*, Chapter II.

†A. Cauchy, *Course d'Analyse*, Chapter XII, §1, Theorem I.

‡Legendre, *Theorie des Numbers*, 1re Part, §XIV; In the German translation by Maser, §119.

As matters now stand the several branches of vector analysis have each a separate notation. Thus there is the S and V notation of quaternions, the hooked bracket and sign of the complement of Grassmann's system, and Professor Gibbs's dot and cross, all denoting, if not always the same concepts, at least the same numerical quantities. Professor Gibbs's notation is distinctly an innovation and marks a tendency away from uniformity.

It is easy to infer why Professor Gibbs desired to throw overboard the quaternion notation. In the first place he desired juxtaposition to denote his so-called indeterminate products and then he wanted the places before and after his dyads unoccupied so that any new sign of operation desired could be inserted. This the quaternion S and V notation did not admit of. Again Professor Gibbs regards S and V as product rather than function symbols so that writing Sab and Vab is to him like writing $\times$ab for a$\times$b. It is evident from this that the quaternion notation lacks symmetry. But there is still another objection to S and V as operation symbols. They are large and for that reason do not look well, nor can they be made quickly. Presumably it was for these reasons that Professor Gibbs gave up the quaternion notation entirely and introduced the dot and cross, the first to take the place of S and the latter of V.

Professor Gibbs has used a marked improvement in the general notation for vector analysis along another line in employing Clarendon type to denote vectors and the same letters in italics to denote their tensors. He thus avoids the use of Hamilton's cumbrous T and U. His system however is open to serious criticism. Besides being out of harmony with the quaternion symbols, this notation contains an important defect in that it uses up all of the signs available to signify multiplication, viz., juxtaposition, the dot, and the cross. Now there are other forms of product formation besides the dyadic and scalar and vector products, as for instance, the quaternion itself, or a product in which enters a trigonometrical function other than the sine and cosine. To have a system of notation to denote other forms of product, one would be under the necessity of inventing for them new arbitrary marks, which course would be open to serious objection. Fortunately all interests can be harmonized and ends met by the simple expedient of using small *letters* for the operation symbols and writing them *between* their factors.

The complete notation proposed may be described very briefly as the use of capitals to denote unit quantities, Clarendon type for vector quantities, the corresponding letters in italics for their tensors, and small letters, preferably initials, written a little above the line of writing, as symbols of operation, with the single exception of the indeterminate product, where no sign is used. Of course between two scalars or between a scalar and a vector no sign means multiplication. Thus ab=dyad or indeterminate product of vectors a and b.

a ^{s} b=Sab=scalar product of vectors a and b.

a ^{V} b=Vab=vector product of vectors a and b.

a ^{t} b=scalar product of tensors of a and b times tangent of included angle.

$a q\, b$=quaternion product of vectors a and b.

$a g\, b$, or [ab]=Grassmann outer product of vectors a and b.

The writer has found this notation as convenient as the dot and cross. It is commended to the student of vector analysis as practical and as retaining the original Hamilton symbols. Evidently little effort is needed to pass from this to the old S, V, T, U notation. Finally, it is adapted to the needs of the general science of vector analysis and admits freely of the introduction of new products.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

167. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

A traveler notices that $m=2\frac{1}{2}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of the train in miles per hour. How far apart are the poles?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let n=number of spaces passed per minute, d=distance between poles.

Then mn miles per hour$=\dfrac{mn\times5280}{60}=88mn$ feet per minute.

$\therefore$ $dn=88mn$, or $d=88m$ feet. When $m=2\frac{1}{2}$, $d=220$ feet.

168. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

If I buy for $m=10\%$ and $n=5\%$ *less* I shall gain $p=15\%$ and $q=5\%$ *more*. What is my rate of gain?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This problem is not clear. I interpret it to mean: If I buy for $100\times90\times95=85.5\%$ instead of for 100%, and sell for the same price as I sold when I paid 100%, I would gain $100\times115\times105=120\frac{3}{4}\%$, $-100\%=20\frac{3}{4}\%$ more than I did gain. $85\frac{1}{2}\%=\frac{171}{200}$, $20\frac{3}{4}\%\frac{83}{400}$, $\frac{83}{400}+1=\frac{483}{400}$. $(\frac{483}{400}\times171-200)\div(200-171=(\frac{82593}{400}-200)\div(200-171)=6\frac{193}{400}\div29=\frac{2593}{11600}$, $\frac{2593}{11600}$ of $100\%=22\frac{41}{116}\%$.

ALGEBRA.

175. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the conditions that $\frac{x}{m+3}+\frac{y}{m+1}+\frac{z}{m-z}=1$, where m may be a, b or c.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x=(m+3)u$, $y=(m-1)v$, $z=(m-z)w$.

$\therefore u+v+w=1$. $\therefore u, v, w$ are the areal coördinates of a point. Let d, e, f be the sides of the triangle of reference; then

$$\frac{u}{d\alpha}=\frac{v}{e\beta}=\frac{w}{f\gamma}=\frac{1}{2\triangle},$$

where $\triangle$ =area of triangle of reference, and $d\alpha+e\beta+f\gamma=2\triangle$.

$$\therefore u=d\alpha/2\triangle,\ \therefore x=\frac{(m+3)d\alpha}{2\triangle};$$

$$v=e\beta/2\triangle,\ \therefore y=\frac{(m-1)e\beta}{2\triangle};$$

$$w=f\gamma/2\triangle,\ \therefore z=\frac{(m-z)f\gamma}{2\triangle},\ \text{or}\ z=\frac{mf\gamma}{2\triangle+f\gamma}.$$

$$\therefore \frac{x}{m+3}+\frac{y}{m-1}+\frac{z}{m-z}=\frac{d\alpha+e\beta+f\gamma}{2\triangle}=\frac{2\ \triangle}{2\ \triangle}=1,\ \text{whatever the value of}\ m.$$

176. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve $x^2+y^2+z=a$.....(1), $x+y^2+z^2=b$.....(2), $x^2+y+z^2=c$.....(3).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2+y^2+z^2=s$.

$\therefore z^2-z=s-a$, or $z=\frac{1}{2}\pm\sqrt{(s-a+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-a)}$.

$x^2-x=s-b$, or $x=\frac{1}{2}\pm\sqrt{(s-b+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-b)}$.

$y^2-y=s-c$, or $y=\frac{1}{2}\pm\sqrt{(s-c+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-c)}$.

$\therefore x^2+y^2+z^2=s=\frac{3}{4}\pm[\sqrt{(m-a)}+\sqrt{(m-b)}+\sqrt{(m-c)}]+3m-(a+b+c)$.

$\therefore 2m+1-(a+b+c)=\mp[\sqrt{(m-a)}+\sqrt{(m-b)}+\sqrt{(m-c)}]$.

$\therefore 2m+D\pm\sqrt{(m-a)}=\mp[\sqrt{(m-b)}+\sqrt{(m-c)}]$.....(1). Squaring (1),

$4m^2+(4D-1)m+D^2+b+c-a=2\{\sqrt{[(m-b)(m-c)]}$

$\mp(2m+D)\sqrt{(m-a)}]$.....(2).

Squaring (2) and remembering that $a+b+c=1-D$,

$$16m^4+8(4D-3)m^3+(24D^2-32D+5)m^2+(8D^3-14D^2+6D+2)m$$
$$+2D^2-2D^3-2(ab+ac+bc)=\mp 8(2m+8)\sqrt{[(m-a)(m-b)(m-c)]},$$

or $16m^4+Am^3+Bm^2+Cm+E=\mp 8(2m+8)\sqrt{[(m-a)(m-b)(m-c)]}$.....(3).

Squaring (3), $256m^8+32Am^7+(A^2+32B)m^6+(32C+2AB-256)m^5$

$$+[B^2+32C+2AC-2048+256(a+b+c)]m^4+[2AC+2BC$$
$$-4096+2048(a+b+c)-256(ab+ac+bc)]m^3+[C^2+2BE$$
$$+4096(a+b+c)-2048(ab+bc+ac)+256abc]m^2+[2CE$$
$$-4096(ab+ac+bc)+2048abc]m+E^2+4096abc=0.$$

This equation gives m and hence s, which finally gives x, y, z.

Solved in a similar manner by the *PROPOSER*.

177. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $m^{2x}(m^2+1)=(m^{3x}+m^x)m$.

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.; CHARLES E. BASSETT, Central University, Danville, Ky.; and E. L. SHERWOOD, Shady Side Academy, Pittsburg, Pa.

The equation may be written $\dfrac{m^{3x}+m^x}{m^{2x}}=\dfrac{m^2+1}{m}$, or $m^x+\dfrac{1}{m^x}=m+\dfrac{1}{m}$,

whence $m^{2x}-\left(m+\dfrac{1}{m}\right)m^x=-1$, from which $m^x=m$ and $1/m$.

Therefore $m^{x\pm1}=1$ and $x\pm1=0$; $x=+1$ and -1.

Also solved by *G. W. GREENWOOD*, and *G. B. M. ZERR*. Professor Zerr finds by performing the indicated operations and factoring, in addition to the roots given above, the root $-\infty$.

GEOMETRY.

197. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Two points P_1, Q_1 are on a generator of a hyperboloid, and P_2, Q_2 the corresponding points on a confocal hyperboloid. Prove $P_1Q_1=P_2Q_2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/a^2-y^2/b^2-z^2/c^2=x^2/\alpha^2-y^2/\beta^2-z^2/\gamma^2=1$ be the hyperboloid and its confocal;

P_1, $Q_1=(d, e, f)$, (h, k, l);

P_2, $Q_2=(m, n, p)$, (r, s, t). We are to prove,

$$(d-h)^2+(e-k)^2+(f-l)^2=(m-r)^2+(n-s)^2+(p-t)^2.$$

For corresponding points,

$$d/a=m/\alpha,\ e/b=n/\beta,\ f/c=p/\gamma,\ h/a=r/\alpha,\ k/b=s/\beta,\ l/c=t/\gamma.$$

$$\therefore\ [(ma/\alpha)-h]^2+[(nb/\beta)-k]^2+[(pc/\gamma)-l]^2$$
$$=[m-(h\alpha/a)]^2+[n-(k\beta/b)]^2+[p-(l\gamma/c)]^2,$$

or $(m/\alpha-h/a)^2(a^2-\alpha^2)+(n/b-k/b)^2(b^2-\beta^2)+(p/\gamma-l/c)^2(c^2-\beta^2)=0.$

This is clearly the case since

$$h^2/a^2-k^2/b^2-l^2/c^2=m^2/\alpha^2-n^2/\beta^2-p^2/\gamma^2=1,$$
$$\text{or } m^2/a^2-h^2/a^2-(n^2/\beta^2-k^2/b^2)-(p^2/\gamma^2-l^2/c^2)=0.$$

$$\therefore\ (m/\alpha-h/a)(m/\alpha+h/a)-(n/\beta-k/b)(n/\beta+k/b)-(p/\gamma-l/c)(p/\gamma+l/c)=0.$$

This is the case when $m/\alpha=h/a$, $n/\beta=k/b$, $p/\gamma=l/c$.

198. Proposed by **JOHN J. QUINN**, Professor of Mathematics, Warren High School, Warren, Pa.

Trisect an angle (1) by means of the cissoid; (2) by means of the paroboloid.

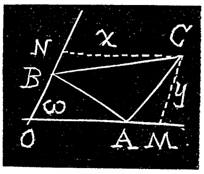
No solution of this problem has been received.

199. Proposed by **F. ANDEREGG, A. M.**, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Two vertices of a given triangle move along fixed right lines; find the locus of the third vertex. [From Salmon's Conics, Sixth Edition, page 208, Ex. 10.]

Solution by **M. W. HASKELL**, Professor of Mathematics, The University of California, Berkeley, Cal.

Let us take the fixed lines as the coördinate axes, and denote the angle between them by ω; the interior angles of the triangle by A, B, C, and the lengths of the opposite sides by a, b, c, respectively; and let φ denote the variable angle OAB. From the triangles AMC, BNC, we have immediately from the theorem of sines



$$y/b=\frac{\sin(\pi-A-\varphi)}{\sin(\pi-\omega)}=\frac{\sin(A+\varphi)}{\sin\omega};\quad x/a=\frac{\sin(\omega-B+\varphi)}{\sin\omega}$$

which may immediately be rewritten in the form

$$\sin A\cos\varphi+\cos A\sin\varphi=(y/b)\sin\omega,$$
$$\sin(\omega-B)\cos\varphi+\cos(\omega-B)\sin\varphi=(x/a)\sin\omega.$$

Solving these equations for $\cos\varphi$ and $\sin\varphi$, and observing that $\sin(A+B-\omega)=\sin(C+\omega)$, we have

$$\cos\varphi\sin(C+\omega)=\sin\omega[-(x/a)\cos A+(y/b)\cos(\omega-B)]$$
$$\sin\varphi\sin(C+\omega)=\sin\omega[(x/a)\sin A-(y/b)\sin(\omega-B)].$$

Squaring and adding, and transposing the members of the equation, we obtain the required equation of the locus

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\cos(C+\omega)+\frac{y^2}{b^2}=\frac{\sin^2(C+\omega)}{\sin^2\omega}.$$

The locus is therefore an ellipse with center at the intersection of the two lines.

Solved similarly by *G. W. GREENWOOD*, and *G. B. M. ZERR*.

CALCULUS.

163. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Can there be a plane curve the length of which varies *directly as the abscissa* and *inversely as the ordinate* of any point on the curve ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$S=mx/y,\ dS=\frac{mydx-mxdy}{y^2}=\sqrt{(dx^2+dy^2)}.$$

$\therefore \sqrt{[1+(dx/dy)^2]}=[my(dx/dy)-mx]/y^2.$
Let $x=vy$, $dx/dy=v+y(dv/dy)=v+yp$, suppose.
$\therefore \sqrt{[1+(v+yp)^2]}=mp$, or $v+yp=\sqrt{(m^2p^2-1)}$.....(1).

From (1), $p=\dfrac{vy\pm\sqrt{(v^2m^2+m^2-y^2)}}{m^2-y^2}=\dfrac{xy\pm\sqrt{[m^2(x^2+y^2)-y^4]}}{y(m^2-y^2)}.$

Differentiating (1) with respect to y,

$$dv/dy=p=p--y(dp/dy)+\frac{m^2p(dp/dy)}{\sqrt{(m^2p^2-1)}}.$$

$$\therefore 2p=\left(\frac{m^2p}{\sqrt{(m^2p^2-1)}}-y\right)\frac{dp}{dy}.$$

$$\therefore \frac{dy}{dp}=\frac{m^2}{2\sqrt{(m^2p^2-1)}}-\frac{y}{2p} \text{ or } \frac{dy}{dp}+\frac{y}{2p}=\frac{m^2}{2\sqrt{(m^2p^2-1)}}.$$

$$\therefore y\sqrt{p}=C+\frac{m^2}{2}\int\frac{\sqrt{(p)}dp}{\sqrt{(m^2p^2-1)}}=C+\frac{m^2}{2}f(p).....(2).$$

The value of p in (2) from (1) gives the primitive, which is the curve desired. By series we get

$$\frac{m^2}{2}\int\frac{\sqrt{(p)}dp}{\sqrt{(m^2p^2-1)}}=\frac{1}{2\sqrt{p}}\sqrt{(m^2p^2-1)}+\tfrac{1}{2}m\sqrt{(p)}(1+\frac{1}{2.3m^2p^2}+\frac{1}{2.4.7m^4p^4}$$

$$+\frac{3}{2.4.6.11m^6p^6}+\frac{3.5}{2.4.6.8.15m^8p^8}+\ldots\ldots).$$

164. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $m^2+n^2=1$, $m^2\cos^2\theta+n^2\cos^2\varphi=A$, $a^2b^2\sin^2\theta(m^2+n^2\cos^2\varphi)+a^2c^2\cos^2\theta\cos^2\varphi$ $+b^2c^2\sin^2\varphi(n^2+m^2\cos^2\theta)=B$, $\sqrt{(1-m^2\sin^2\theta)}=\triangle(\theta)$, $\sqrt{(1-n^2\sin^2\varphi)}=\triangle(\varphi)$ prove that $\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\triangle(\varphi)}=(\frac{1}{6}\pi)(a^2b^2+a^2c^2+b^2c^2)$.

Solution by the PROPOSER.

$x^2/a^2+y^2/b^2+z^2/c^2=1$.

$$\int ds=\int\int\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}dxdy=c^2\int\int\frac{\sqrt{[z^2/c^4+x^2/a^4+y^2/b^4]}dxdy}{z}$$

Let p=perpendicular from center on tangent plane.

$\therefore\ 1/p=\sqrt{[z^2/c^4+x^2/a^4+y^2/b^4]}$.

Let $(b/a)\sqrt{[a^2-x^2]}=y'$, $(a/b)\sqrt{[b^2-y^2]}=x'$.

$$D=\int\frac{ds}{p}=c^2\int\int\frac{[z^2/c^4+x^2/a^4+y^2/b^4]dxdy}{z}=\frac{1}{c^2}\int_0^a\int_0^{y'}zdxdy+\frac{c^2}{a^4}\int_0^a\int_0^{y'}\frac{x^2}{z}dxdy$$

$$+\frac{c^2}{b^4}\int_0^b\int_0^{x'}(y^2/z)dydx=\frac{1}{abc}\int_0^a\int_0^{y'}\sqrt{[a^2b^2-b^2x^2-a^2y^2]}dxdy$$

$$+\frac{abc}{a^4}\int_0^a\int_0^{y'}\frac{x^2dxdy}{\sqrt{[a^2b^2-b^2x^2-a^2y^2]}}+\frac{abc}{b^4}\int_0^b\int_0^{x'}\frac{y^2dydx}{\sqrt{[a^2b^2-b^2x^2-a^2y^2]}}$$

$$=\tfrac{1}{6}\pi(ab/c+bc/a+ac/b)=\frac{\pi}{6abc}(a^2b^2+b^2c^2+a^2c^2).$$

Let $x=a\sin\varphi\sqrt{(1-m^2\sin^2\theta)}$, $y=b\cos\theta\cos\varphi$, $z=c\sin\theta\sqrt{(1-n^2\sin^2\varphi)}$,

$m^2+n^2=1$; $dx/d\varphi=a\cos\varphi\sqrt{(1-m^2\sin^2\theta)}$, $dx/d\theta=-\dfrac{am^2\sin\varphi\sin\theta\cos\theta}{\sqrt{(1-m^2\sin^2\theta)}}$, $dy/d\varphi=$ $-b\cos\theta\sin\varphi$, $dy/d\theta=-b\sin\theta\cos\varphi$; $dxdy=(dx/d\theta)(dy/d\varphi)-(dx/d\varphi)(dy/d\theta)$

$$=\frac{ab}{\sqrt{(1-m^2\sin^2\theta)}}(m^2\sin^2\varphi\sin\theta\cos^2\theta+\sin\theta\cos^2\varphi-m^2\sin^3\theta\cos^2\varphi)=\frac{Aab\sin\theta}{\triangle(\theta)}.$$

$$\frac{c^2}{z}(x^2/a^4+y^2/b^4+z^2/c^4)=\frac{c}{\sin\theta\triangle(\varphi)}\Big(\frac{\sin^2\varphi(1-m^2\sin^2\theta)}{a^2}+\frac{\cos^2\theta\cos^2\varphi}{b^2}$$

$$+\frac{\sin^2\theta(1-n^2\sin^2\varphi)}{c^2}\Big)=\frac{1}{a^2b^2c\sin\theta\triangle(\varphi)}[b^2c^2\sin^2\varphi(n^2+m^2\cos^2\theta)$$

$$+a^2c^2\cos^2\theta\cos^2\varphi+a^2b^2\sin^2\theta(m^2+n^2\cos^2\varphi)]=\frac{B}{a^2b^2c\sin\theta\triangle(\varphi)}.$$

$$\therefore \int\frac{ds}{p}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{abc\triangle(\theta)\triangle(\varphi)}=\frac{\pi}{6abc}(a^2b^2+a^2c^2+b^2c^2).$$

$$\therefore \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\triangle(\varphi)}=\tfrac{1}{6}\pi(a^2b^2+a^2c^2+b^2c^2).$$

Also $\displaystyle\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\triangle(\varphi)}=\tfrac{1}{3}(a^2b^2+a^2c^2+b^2c^2)[E(m)F(n)$

$+F(m)E(n)-F(m)F(n)]$.

$\therefore\ E(m)F(n)+F(m)E(n)-F(m)F(n)=\frac{1}{2}\pi$, Legendre's Theorem.

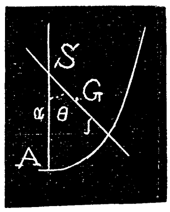
MECHANICS.

155. Proposed by M. E. GRABER. Graduate Student, Heidelberg University, Tiffin, Ohio.

A parabolic curve is placed in a vertical plane with its axis vertical aud vertex downwards, and inside of it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests. Find the position of equilibrium.

Solution by G. W. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill., and E. L. SHERWOOD, Shady Side Academy, Pittsburg, Pa.

Let the equation to the parabola be $2a/r=1+\cos\theta$, and let the length of the rod be $2l$. We must have $2l>a$. When the rod makes an angle θ with the axis, the depth of its center of gravity below the focus is



$$n=(r-l)\cos\theta=\Big(\frac{2a}{1+\cos\theta}-l\Big)\cos\theta.$$

In a position of equilibrium, n is a maximum or a minimum.

$$\therefore\ \sin\theta\Big(\frac{2a}{(1+\cos\theta)^2}-l\Big)=0.\quad \therefore\ \theta=0,\text{ or }\cos\theta=\sqrt{(2a/l)}-1.$$

We must take the positive sign with the radical.

Since $2l>a$, $\sqrt{(2a/l)}<2$, and the values of θ obtained from the last equation are real, but in order that θ be not greater than 90°, an obvious requirement in this problem, we must have $l<2a$. In any case, the position given by $\theta=0$ is unstable. The two values of θ given by the second equation give the positions of stable equilibrium when $2a>l>\frac{1}{2}a$.

Also solved by *G. B. M. ZERR.*

156. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

Three perfectly elastic particles start from the cusp of a smooth cycloid (axis vertical, vertex down) at intervals of t seconds. How long will it be to the nth collision ?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Since the curve is a smooth cycloid and the particles perfectly elastic and of equal mass, whenever collision takes place the two particles that collide have the same velocity. The reaction of one upon the other from impact is the same as though one particle passed through the other without either being affected.

Let mt $(m>2)$ be the time it takes a particle to go from cusp to cusp, and a, the radius of the generating circle. Then $m=(2\pi/t)\sqrt{(a/g)}$. After the first particle has arrived at the opposite cusp, the second is t and the third $2t$ seconds behind it.

$\therefore$ Time to first collision is $mt+t/2=(2m+1)t/2$ seconds.
Time to second collision is $mt+t=(2m+2)t/2$ seconds.
Time to third collision is $mt+3t/2=(2m+3)t/2$ seconds.
Time to fourth collision is $mt+t+mt-t+t/2=(4m+1)t/2$ seconds.
Time of fifth collision is $(4m+2)t/2$ seconds.
Time of sixth collision is $(4m+3)t/2$ seconds; etc.

$\therefore$ The collisions take place in sets of three. The pth set is $(2pm+1)t/2$, $(2pm+2)t/2$, $(2pm+3)t/2$. If the nth collision is the 1st, 4th, 7th, ...$(3p-2)$th, then $n=3p-2$ or $p=(n+2)/3$, and it takes place in $[2(n+2)m+3]t/6$ seconds; if it is the 2nd, 5th, 8th, ...$(3p-1)$th, it takes place in $[2(n+1)m+6]t/6$ seconds; if it is the 3rd, 6th, 9th, ...$3p$th, it takes place in $(2nm+9)t/6$ seconds.

Also solved by *G. W. GREENWOOD.*

157. Proposed by T. W. WRIGHT, Professor of Mathematics, Schenectady, N. Y.

Explain why a waterfall h feet high can support a column $2h$ feet high.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let w=weight of a cubic foot of water. A column of water 1 foot square falling through h feet weighs hw pounds; to this must be added the kinetic energy of a cubic foot of water falling through h feet; this equals $wv^2/2g$, but $v^2=2gh$.

$\therefore$ kinetic energy$=wh$. $\therefore$ $wh+wh=2wh=w(2h)$.

158. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are electrical images of each other in respect to a given sphere may be constant, and that their centers of gravity shall lie on the surface of the sphere.

Remark by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This problem is the same as problem 148, a solution of which is given in No. 2, Vol. X, pages 48 and 49.

DIOPHANTINE ANALYSIS.

105. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that every factor of $a^{2m}+b^{2m}$ is of the form $1[\text{mod.}2^{m+1}]$ where a and b are prime to each other.

Note by the PROPOSER.

The enunciation of this proposition is corrected on page 207, Vol. IX, of the MONTHLY, and it may be stated in full as follows:

If a, b and m are real positive integers, a and b relatively prime, then every odd factor of $a^{2m}+b^{2m}$ has the form $1\{mod.2^{m+1}\}$.

This special arithmetical theorem was first given by Euler, and may be directly proved by the principles in the theory of power residues. It is interesting, however, to examine its connection with a more general proposition. If ω_1, ω_2.....$\omega_{\phi(n)}$ are all the distinct primitive roots of $x^n-1=0$ and $\phi(n)$ the number of integers less than the positive integer n, and prime to it, then the rational function

$$F\{a,\ b,\ n\}=\{a-\omega_1 b\}\{a-\omega_2 b\}.....\{a-\omega_{\phi(n)}b\}$$

possesses the following property:

Every divisor of $F\{a,\ b,\ n\}$ which is prime to n, has the form $1\{mod\ n\}$.

A proof of this for the case $b=1$, is to be found in Kronecker's *Vorlesungen über Zahlentheorie,*" Vol. I, pages 440-441, and the analysis used may be readily extended to cover the case in which b is general. For $n=2^{m+1}$ (m, a positive integer) it will be noticed that

$$F\{a,\ b,\ 2^{m+1}\}=a^{2m}+b^{2m},$$

and by the general theorem above stated, it follows that every odd divisor of this function has the form $1(\text{mod}2^{m+1})$. When $a=2$, $b=1$, we may write

$$f\{m\}=2^{2^m}+1,$$

and this form gives the remarkable numbers of Gauss which occur in the partition of the perigon.

I notice that the determination of factors of $f\{m\}$ for $m=9$, 11, 12, 18 and 38, has been recently effected by several English mathematicians. For information regarding these interesting results, see the *Educational Times*, June, 1903, page 270, where reference is made to the announcement by Col. J. A. Cunningham. A summary of the previous work done along this line in given in Ball's *Recreations and Problems*, third edition, pages 34-35.

112. Proposed by L. C. WALKER, A. M., Graduate Student Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the nth triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and the PROPOSER.

Let $2x_n-1$, $2x_n$, and $2x_n+1$ denote the sides of the nth triangle, r_n the radius of the inscribed circle, and A the area of the triangle.

Then $A=\sqrt{\{[3x_n(x_n+1)(x_n)(x_n-1)]\}}=3x_nr_n$,

$$\text{whence } [x_n-r_n\sqrt{3}][x_n+r_n\sqrt{3}]=1.....[1].$$

Now for the first triangle, $x=2$, and $r=1$; $\therefore [2-\sqrt{3})^n[2+\sqrt{3}]^n=1.....[2]$.

Assume $x_n-r_n\sqrt{3}=[2-\sqrt{3}]^n$; then from [1], $x_n+r_n\sqrt{3}=[2+\sqrt{3}]^n$; from which $x_n=r_n\sqrt{3}=\frac{1}{2}\{[2+\sqrt{3}]^n+[2-\sqrt{3}]^n\}$. Hence,

$$2x_n-1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n-1,$$

$$2x_n=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n,$$

$$2x_n+1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n+1,$$

$$A=[\tfrac{1}{4}\sqrt{3}]\{[2+\sqrt{3}]^{2n}-[2-\sqrt{3}]^{2n}\}.$$

II. Solution bp A. H. BELL, Litchfield, Ill.

Take the sides of the rational triangle $x+d$, x, $x-d$; the area of this triangle is $\frac{1}{4}x[3(x^2-4d^2)]^{\frac{1}{2}}$. To have it rational we must have $3[x^2-4d^2]=\square=y^2$, say, in which the area is $\frac{1}{4}xy$. If we make $x=2x'$, and $y=6y'$, the equation becomes, when $d=1$, $x'^2-3y'^2=1.....(2)$. The values of x, by Lagrange, are the numerators of the odd convergents from the $\sqrt{3}$; the first two odd fractions are $\frac{1}{0}$, $\frac{2}{1}$, etc., in which $2x'=x=4$, giving the initial rational triangle with sides 3, 4, 5.

The extension of the values of x', or x, by the algebraic solution of this equation in general, is known to be $x_{n+1}=2x.x_n-x_{n-1}.....(4)$.

This constant factor $2x$, is called by Roberts, and Robins, the magic M, or Key. In this case $M=4$, and the middle side of the next rational triangle is by (4), $x_2=4.4-2=14$, and the triangle 13, 14, 15; by extending the middle side in the same way we get,

No. of triangle:	1	2	3	4	5	6	
	3	13	51	193	723	2701	
Sides:	4	14	52	194	724	2702	etc......(5).
	5	15	53	195	725	2703	

The nth rational triangle can be found by entending formula (4) in terms of M, and x', as follows:

No. 0 1 2 3 4

Series $x'=1$; x'; $Mx'-1$; $[M^2-1]x'-M$; $[M^3-2m]x'-M^2+1$;

5 etc.

$[M^4-3M^2+1]x'-M^3+2M$; etc......(6).

By restoring the value of x', above$=2$, we have the series of values of the nth, middle side of the nth rational triangle in terms of the powers of magic M. The quantity in the powers of M multiplied by the constant factor x', I call the "Continued Fraction Theorem" the laws of which are easily determined. These quantities are shown above in brackets in (6), and abbreviated to CFT.

Hence the middle side of the nth triangle$=\{[n\text{th.CFT}]2-(n-1]\text{th.CFT}\}2$.....(7).

The laws of this theorem are: The 1st coefficient of $M=1$.

The 2nd coefficient of $M=$no. cycle -2.

The pth coefficient of $M=$(where $n-p=m$)

$$p\text{th coefficient}=\frac{m[m-1][m-2].....[m-p+2]}{1.2.3.....[p-1]}..... (8).$$

If $n=10$, the 10th.CFT$=[M^9-8M^7+21M^5-20M^3+5M]$, and multiplied by $\frac{1}{2}M=x'=2$, from which take 9th.CFT$=M^8-7M^6+15M^4-10M^2+1$, the result multiplied by 2 will give the middle side, x, of the 10th rational triangle$=524174$.

Then the 10th triangle has for sides 524173, 524174, 524175.

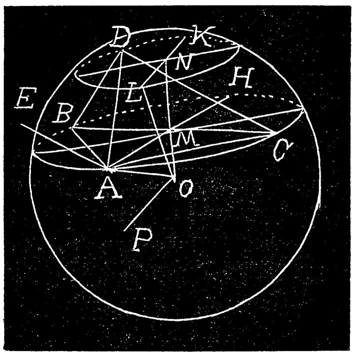
AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three by three, is 1-35 of the voume of the given sphere.

II. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B, C, D be the four random points; AH the diameter of the section through A, B, C; LK the diameter of a section through D parallel to the section through A, B, C; O the center of the sphere, M the center of AH, N the center of LK. Draw AE perpendicular to AH, and OP a line such that AB is parallel to the plane MOP. Let $AO=r$, $\angle AOM=\theta$, $\angle EAC=\varphi$, $\angle EAB=\psi$, $\angle LON=\beta$, $\angle MOP=\lambda$, the angle POM makes with some fixed plane$=\rho$.



Then $AC=2r\sin\theta\sin\varphi$, $AB=2r\sin\theta\sin\psi$; the volume, V, of the tetrahedron $DABC=\frac{2}{3}r^3(\cos\beta-\cos\theta)\sin^2\theta\sin\varphi\sin\psi\sin(\varphi-\psi)$. An element of surface at A is $2\pi r^2\sin\theta d\theta$; at D, $2\pi r^2\sin\beta d\beta$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\lambda\sin\psi d\psi d\rho$. The limits of θ are 0 and π; of β, 0 and θ; of φ, 0 and π; of ψ, 0 and φ and doubled; of λ, 0 and π; of ρ, 0 and 2π. Since the whole number of ways the four points can be taken is $256\pi^4r^8$, we have

$$\triangle=\frac{2}{256\pi^4r^8}\int_0^\pi\int_0^\theta\int_0^\pi\int_0^\phi\int_0^\pi\int_0^{2\pi}V.2\pi r^2\sin\theta d\theta.2\pi r^2\sin\beta d\beta.4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\lambda\sin\psi d\psi d\rho$$

$$=\frac{2r^3}{3\pi}\int_0^\pi\int_0^\theta\int_0^\pi\int_0^\phi\int_0^\pi\sin^5\theta(\cos\beta-\cos\theta)\sin\beta\sin^2\varphi\sin^2\psi\sin^2(\varphi-\psi)\sin\lambda d\theta d\beta$$

$$\times d\varphi d\psi d\lambda$$

$$=\frac{4r^3}{3\pi}\int_0^\pi\int_0^\theta\int_0^\pi\int_0^\phi\sin^5\theta\sin\beta(\cos\beta-\cos\theta)\sin^2\varphi\sin^2\psi\sin^2(\varphi-\psi)d\theta d\beta d\varphi d\psi$$

$$=\frac{r^3}{6\pi}\int_0^\pi\int_0^\theta\int_0^\pi\sin^5\theta\sin\beta(\cos\beta-\cos\theta)(3\varphi\sin^2\varphi-2\varphi\sin^4\varphi-3\sin^5\varphi\cos\varphi)d\theta d\beta d\varphi$$

$$=\frac{r^3}{16\pi}\int_0^\pi\int_0^\theta\sin^5\theta\sin\beta(\cos\beta-\cos\theta)d\theta d\beta$$

$$=\frac{\pi r^3}{32}\int_0^\pi\sin^5\theta(2-2\cos\theta-\sin^2\theta)d\theta=(4\pi r^3/105)=\tfrac{1}{35}\text{ of volume of sphere.}$$

133. **Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.**

A circle of unknown radius is drawn with its center at the vertex of a given parabola, and has its greatest area when its circumference passes through the focus of the parabola. Required the average area common to the circle and parabola.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $y^2=4ax$, $x^2+y^2=r^2$ be the equation to the parabola and circle, respectively, $\triangle$=average area, $x'=\sqrt{(r^2+4a^2)}-2a$.

$$\therefore\triangle=2\int_0^a\left[2\int_0^{x'}\sqrt{(ax)}dx+\int_{x'}^r\sqrt{(r^2-x^2)}dx\right]dr\Big/\int_0^a dr$$

$$=\int_0^a\left[2\int_0^{x'}\sqrt{(ax)}dx+\int_{x'}^r\sqrt{(r^2-x^2)}dx\right]dr=\frac{1}{6a}\int_0^a\Big[3\pi r^2+4\sqrt{(a)}[\sqrt{(r^2+4a^2)}$$

$$-2a]^{\frac{3}{2}}-6r^2\sin^{-1}\left(\frac{\sqrt{(r^2+4a^2)}-2a}{r}\right)\Big]dr=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)$$

$$+\frac{2\sqrt{a}}{3a}\int_0^a[\sqrt{(r^2+4a^2)}-2a]^{\frac{3}{2}}dr+\frac{\sqrt{a}}{3a}\int_0^a\frac{r^2[\sqrt{(r^2+4a^2)}-2a)^{\frac{1}{2}}dr}{\sqrt{(r^2+4a^2)}}.$$

Let $\sqrt{(r^2+4a^2)}=2y-r,\ \frac{a}{2}(\sqrt{5}+1)=y'$.

$$\therefore\ \triangle=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\frac{\sqrt{a}}{3a}\int_a^{y'}\frac{(3y^2+3a^2+2ay)(y-a)^3dy}{y^3\sqrt{y}}$$

$$=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\frac{\sqrt{a}}{3a}\int_a^{y'}(3y\sqrt{y}-7a\sqrt{y}+6a^2/\sqrt{y}-6a^3/y\sqrt{y}$$

$$+7a^4/y^2\sqrt{y}-3a^5/y^3\sqrt{y})dy=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\tfrac{2}{45}a^2[(86-13\sqrt{5})$$

$$\sqrt{\frac{\sqrt{5}+1}{2}}-128+11(11-2\sqrt{5})\sqrt{\frac{\sqrt{5}-1}{2}}].$$

Solved with a different result by the *PROPOSER*.

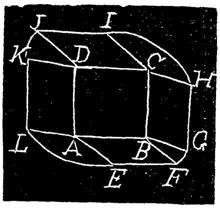
134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

An ellipse, semi-axes a, b, is placed on a square, side c. Find the chance that center of ellipse is on the square.

Solution by the PROPOSER.

Let $ABCD$ be the square, and let the ellipse move parallel to itself so that the major axis makes an angle θ with the side BC. Then the center of the ellipse will describe the area $GHIJKLEFG$. Let $(a^2-b^2)/a^2=e^2$, p=chance. The perpendicular distance from BC to $GH=a\sqrt{(1-e^2\cos^2\theta)}$, the perpendicular distance from DC to $IJ=a\sqrt{(1-e^2\sin^2\theta)}$.

Area$=\pi ab+c^2+2ac\,[\sqrt{(1-e^2\sin^2\theta)}+\sqrt{(1-e^2\cos^2\theta)}]=u$.



$$1/p=\int_0^{\frac{1}{2}\pi}u\,d\theta\Big/\int_0^{2\pi}c^2\,d\theta=\frac{2}{\pi c^2}\int_0^{\frac{1}{2}\pi}u\,d\theta.\quad\therefore\ 1/p=[\pi ab+c^2+\frac{8ac}{\pi}E(e,\tfrac{1}{2}\pi)]/c^2.$$

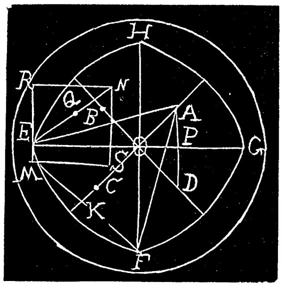
If $a=b$, $1/p=(\pi a^2+c^2+4ac)/c^2$. If $a=b=c$, $1/p=\pi+1+4$.

135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

If the line joining two points taken at random in the surface of a given circle be the diagonal of a square, the chance that the square lies wholly within the circle is $2-4/\pi$.

Solution by the PROPOSER.

Let MN be the line joining the random points M, N; $MRNS$ the square; Q, its center; O, the center of the circle. Let the square move about the circle, so as to be within it, but in contact with it, the diagonal MN remaining parallel



to itself; Q will describe the figure $EFGH$, whose boundary consists of the four equal arcs, EF, FG, GH, HE, whose centers are A, B, C, D, and radius equal to that of the given circle.

Now while MN is given in length and direction, the area of the figure $EFGH$ represents the number of ways the two points can be taken, so that the square will be wholly within the circle.

Let $AE=r$, $MN=2x$, $\angle AEP=\theta$, $\phi=$the angle which MN makes with some fixed line, area $EFGH=u$.

Then we have $OA=x$, $AP=\frac{1}{2}x\sqrt{2}$, $\angle EAK=\angle AOP-\angle AEP=\frac{1}{4}\pi-\theta$, $\frac{1}{2}x\sqrt{2}=r\sin\theta$, $EK=r\sin(\frac{1}{4}\pi-\theta)$, area $EOF=r^2(\frac{1}{4}\pi-\theta)-rx\sin(\frac{1}{4}\pi-\theta)=r^2(\frac{1}{4}\pi-\theta-\sin\theta\cos\theta+\sin^2\theta)$, and $u=r^2(\pi-4\theta-4\sin\theta\cos\theta+4\sin^2\theta)$. An element of the circle at the point N is $4xdxd\phi$, or $8r^2\sin\theta\cos\theta d\theta d\phi$. The limits of θ are 0 and $\frac{1}{4}\pi$; and of ϕ, 0 and 2π. Hence the chance in question is

$$\delta=\frac{1}{\pi^2r^4}\int_0^{\frac{1}{4}\pi}\int_0^{2\pi}r^2(\pi-4\theta-4\sin\theta\cos\theta+\sin^2\theta).8r^2\sin\theta\text{cas}\theta d\theta d\phi$$

$$=\frac{16}{\pi}\int_0^{\frac{1}{4}\pi}(\pi-4\theta-4\sin\theta\cos\theta+4\sin^2\theta)\sin\theta\cos\theta d\theta=2-(4/\pi).$$

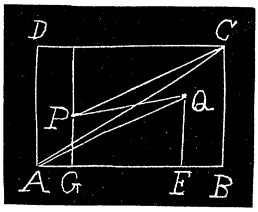
An excellent solution of this problem was also received from *G. B. M. ZERR.*

136. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

By direct computation find the average distance between two points in the surface of a given rectangle, but on the opposite sides of a diagonal.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

Let P, Q be the random points, $AB=a$, $BC=b$, $a^2+b^2=d^2$, $AE=x$, $QE=y$, $CF=u$, $PF=v$. Then $PQ=\sqrt{[(x+u-a)^2+(y+v-b)^2]}$. The limits of x are 0 and a; of u, 0 and a; of y, 0 and bx/a; of v, 0 and bu/a. Let $(x+u-a)=c$, $\triangle=$average distance.



$$\therefore\triangle=\frac{4}{a^2b^2}\int_0^a\int_0^a\int_0^{bx/a}\int_0^{bu/a}\sqrt{[c^2+(y+v-b)^2]}$$

$$\times dxdudydv=\frac{2}{a^2b^2}\int_0^a\int_0^a\int_0^{bx/a}\Big[\Big(\frac{ay+bu-ab}{a^2}\Big)$$

$$\times\sqrt{(a^2c^2+(ay+bu-ab)^2]}-(y-b)\sqrt{[c^2+(y-b)^2]}$$

$$+c^2\log\Big(\frac{ay+bu-ab+\sqrt{[a^2c^2+(ay+bu-ab)^2)}}{ay-ab+a\sqrt{[c^2+(y-b)^2]}}\Big)\Big]dxdudy$$

$$=\frac{2}{3a^5b^2}\int_0^a\int_0^a\{[2a^2c^2-b^2(u-a)^2]\sqrt{[a^2c^2+b^2(u-a)^2]}+[(2a^2c^2-b^2)(x-a)^2]$$

$$\times\sqrt{[a^2c^2+b^2(x-a)^2]}-a^3(2c^2-b^2)\sqrt{[b^2+c^2]}+dc^3(d^2-3a^2)$$

$$+3a^2bc^3\log[c(b+d)]-3a^3bc^2\log a-3a^2bc^2(x-a)\log[b(x-a)$$

$$+\sqrt{[a^2c^2+b^2(x-a)^2]}]-3a^3bc^2\log[\sqrt{(b^2+c^2)}-b]-3a^2bc^2(u-a)\log[b(u-a)$$

$$+\sqrt{[a^2c^2+b^2(u-a)^2]}]\}dxdu$$

$$=\frac{1}{72a^5b^2}\int_0^a\Big[24x[2a^2x^2-3b^3(x-a)^2]\sqrt{[a^2x^2+b^2(x-a)^2]}-16a^3bx^3$$

$$-16a^3b(x-a)^3+(x-a)^4(72b^2d-12a^2d-12d^3-23a^2b)+x^4(12d^3-36a^2d$$

$$+23a^2b)+12a^3x(3b^3-2x^2)\sqrt{[b^2+x^2]}+12a^3(x-a)[2(x-a)^2-3b^2]$$

$$\times\sqrt{[b^2+(x-a)^2]}+60a^2b(x-a)^4\log[(b+d)(x-a)]$$

$$+\frac{24b^4(x-a)^4}{a}\log[(a+d)(x-a)]-96a^2bx^3(x-a)\log\{b(x-a)$$

$$+\sqrt{[a^2x^2+b^2(x-a)^2]}\}-48a^3bx^3\log[\sqrt{(b^2+c^2)}-b]-\frac{24b^4(x-a)^4}{a}$$

$$\times\log\{ax+\sqrt{[a^2x^2+b^2(x-a)^2]}\}+48a^3b(x-a)^3\log\{\sqrt{[b^2+(x-a)^2]}-b\}$$

$$+12a^3b^4\log[x+\sqrt{(b^2+x^2)}]-12a^3b^4\log\{x-a+\sqrt{[b^2+(x-a)^2]}\}$$

$$+36a^2bx^4\log[x(b+d)]-48a^3bx^3\log a+48a^3b(x-a)^3\log a\Big]dx$$

$$=\tfrac{1}{15}\left[2d\left(2-\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)-\frac{a^3+b^3}{d^2}+\frac{2a^3}{b^2}+\frac{2b^3}{a^2}+\left(\frac{4b^2}{a}-\frac{a^2b^2}{d^3}\right)\log\left(\frac{a+d}{b}\right)\right.$$

$$\left.+\left(\frac{4a^2}{b}-\frac{a^2b^2}{d^3}\right)\log\left(\frac{b+d}{a}\right)\right]=\frac{a}{30}[6+(16-\sqrt{2})\log(1+\sqrt{2})],\text{ if } a=b.$$

MISCELLANEOUS.

135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Find invariants of the second, third, and sixth degrees in the coefficients of a binary quartic.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=u=0$, be the binary quartic.

Then $d^4u/dx^4=24a$, $d^4u/dx^3dy=24b$, $d^4u/dx^2dy^2=24c$, $d^4u/dxdy^3=24d$, $d^4u/dy^4=24e$.

Substitute in u, d^4u/dx^4 for y^4, d^4u/dy^4 for x^4, $-d^4u/dx^3dy$ for xy^3, $-d^4u/dxdy^3$ for x^3y, d^4u/dx^2dy^2 for x^2y^2, and we get $24ae-96bd+144c^2-96bd+24ae=48(ae-4bd+3c^2)$, the invariant of second order.

$d^2u/dx^2=12(ax^2+2bxy+cy^2)$.

$d^2u/dy^2=12(cx^2+2dxy+ey^2)$.

$d^2u/dxdy=12(bx^2+2cxy+dy^2)$.

$\therefore$ The Hessian of this quartic is

$$(ax^2+2bxy+cy^2)(cx^2+2dxy+ey^2)-(bx^2+2cxy+dy^2)^2=0, \text{ or } (ac-b^2)x^4$$
$$+2(ad-bc)x^3y+(ae+2bd-3c^2)x^2y^2+2(be-cd)xy^3+(ce-d^2)y^4=0.$$

Write this in the form $Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4=0$.

Then $24(aE-4bD+6cC-4dB+Ae)=72(ace+2bcd-ad^2-eb^2-c^3)$, is the invariant of the third order.

$\therefore S=ae-4bd+3c^2$, is the invariant of the second order.

$T=ace+2bcd-ad^2-eb^2-c^3$ is the invariant of the third order.

These are the only two ordinary invariants. The invariant of the sixth order being the discriminant which we determine as below.

If we had used the more general form $Ax^4+By^4+Cz^4$, where $x+y+z=0$, then we would have had $a=A+C$, $e=B+C$, $b=C=e=d$.

$\therefore S=BC+CA+AB$, $T=ABC$.

Equating to zero the two differential equations Ax^3-Cz^3, and By^3-Cz^3, we get $Ax^3=By^3=Cz^3$, or $ABCx^3/BC=ABCy^3/AC=ABCz^3/AB$.

$\therefore x^3 : y^3 : z^3=BC : AC : AB$, this in $x+y+z=0$ gives $(BC)^{\frac{1}{3}}+(AC)^{\frac{1}{3}}+(AB)^3=0$.

$\therefore (BC+AC+AB)^3-27A^2B^2C^2=0$, or $S^3-27T^2=0$.

$\therefore (ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-eb^2-c^3)^2=$the sixth order.

Also solved by *G. W. GREENWOOD*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

At what rate per cent. must a note be discounted at the end of every quarter of a year in order to produce a discount equivalent to 10% interest for the year?

ALGEBRA.

181. **Proposed by J. F. LAWRENCE, Breckenridge, Mo.**

Show that $\phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-$ *ad inf.* $=\frac{x(1-x^2)}{(1+x^2)^2}$, $\phi(n)$ being the number of integers less than n and prime to it. [From Hall and Knight's *Higher Algebra*, page 358].

182. **Proposed by J. F. LAWRENCE, Breckenridge, Mo.**

Find the values of x_1, x_2, x_3, ...x_n which satisfy the following system of simultaneous equations:

$$\frac{x_1}{a_1-b_1}+\frac{x_2}{a_1-b_2}+.....+\frac{x_n}{a_1-b_n}=1.$$

$$\frac{x_1}{a_2-b_1}-\frac{x_2}{a_2-b_2}+.....+\frac{x_n}{a_2-b_n}=1.$$

$$..... \quad ..... \quad ..... \quad ..... \quad .....$$

$$\frac{x_1}{a_n-b_1}+\frac{x_2}{a_n-b_2}+.....+\frac{x_n}{a_n-b_n}=1.$$

GEOMETRY.

203. **Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.**

Show that two parabolae can always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex is parallel to the opposite side.

204. **Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.**

Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides. [Phillips and Fisher].

CALCULUS.

168. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

The tangent of what Cartesian curve makes an x-intercept always m times as long as the corresponding y-intercept?

169. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

Find the value of y from the Eulerian equation

$$y=\int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}.$$

MECHANICS.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

Required the time for a tree, considered as a material line of uniform density, length $a=100$ feet, to fall; the tree being inclined $\phi=1'$ from perpendicular.

160. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Given the paracentric acceleration c^2/r^4 and the angular velocity $(n/m)\pi$, to determine the equation of the orbit.

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ..., denote all its distinct integral divisors, then $2^n>2(n+1)(n'+1)(n''+1)$.....

117. Proposed by R. W. D. CHRISTIE.

Without the use of the method of continued fractions, solve the equation $x^2-149y^2=1$, and generalize your method. [From the *Educational Times*].

AVERAGE AND PROBABILITY.

145. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

In each quadrant of a given circle, a circle is described at random. A point is taken at random in each of these circles. What is the average area of the quadrilateral formed by joining with straight lines these four points?

146. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A random straight line crosses a given ellipse; find the chance that two points, taken at random in the ellipse, shall lie on opposite sides of the line.

MISCELLANEOUS.

140. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prof. Felix Klein asserts that

$$I_1=\int\frac{dz}{\sqrt{[z(1-z)(1-kz)]}}=2\mathrm{sn}^{-1}(\sqrt{z})$$

is a *more* canonical form of the first Legendrian elliptic integral than

$$I_2=\int_0^x\frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}=\mathrm{sn}^{-1}(x,\kappa).$$

Show this.

141. Proposed by O. W. ANTHONY.

Is f, where f is an operation defined by the equation $f(\mu\nu)=\mu f(\nu)+\nu f(\mu)$, necessarily distributive?

NOTES.

Lewis Neikirk took his Doctor's degree at the University of Pennsylvania on the 17th of June.

Dr. E. R. Hedrick of the Sheffield Scientific School of Yale University, has been elected to the chair of mathematics of the State University of Missouri.

Dr. J. W. Young, who took his Doctor's degree at Cornell this year, has been appointed Assistant Professor of Mathematics at Northwestern University.

Editor Dickson desires to obtain the following back numbers of the MONTHLY to complete his set: Vol. IV, No. 6-7, June-July, 1897; Vol. VI, No. 9-10, October, 1899; Vol. VII, No. 1, January, 1900. Any one having extra copies of one or more of these three numbers will confer a favor by notifying him.

On April 7th, 1903, Editor Finkel was appointed, by the Trustees of the University of Pennsylvania, Special Fellow in Mathematics for the period of one year from the first of September, 1903. In view of this appointment, the Executive Committee of the Board of Trustees of Drury College has granted him a leave of absence for one year. His address, therefore, after September 25th, will be the University of Pennsylvania, Philadelphia, Penn.

In order, therefore, that he may get the greatest good out of the opportunity thus offered for recreation and study, it is desirable that he be relieved as much as possible from the cares and responsibilities of the management of the MONTHLY. It was hoped that a full statement of the arrangement for its management could be made in this issue and the issue was withheld from the mail for over a week awaiting the decision of a gentleman to whom Mr. Finkel's position had been offered. But this gentleman, at the last moment, accepted a permanent position at another place. Hence, at this writing, the matter is not definitely settled. But whatever arrangement is made we wish to assure our readers that the work of the MONTHLY will be carried on, and no pains will be spared on the part of those entrusted to its management to keep up its high standard of efficiency and usefulness.

As during the past year, Dr. Dickson will continue to edit the papers which are offered for publication, and all matters pertaining thereto should be addressed to him at the University of Chicago.

All subscriptions for the coming year should be sent to Mr. W. C. Calland, Treasurer of Drury College, and also all requests for sample copies or missing numbers should be addressed to him.

BOOKS AND PERIODICALS.

The Principles of Mathematics. By Bertrand Russell, A. M., late Fellow of Trinity College, Cambridge. Vol. I., large 8 vo. Cloth. xxix+534 pages. Price $3.50 New York: The Macmillan Co.

The author says that the object of the work is two fold. One of these objects is to prove that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of logical principles. This proof is undertaken in this volume in parts II—VII. In volume II, a demonstration of the same proposition is to be given by strict symbolic reasoning.

The second object of the work, occupying part I, is the explanation of the fundamental concepts which mathematics accepts as indefinable.

The work before us which is addressed in equal measure to philosophers and mathematicians, must not be considered easy reading. Indeed, some chapters, as for example, the chapter on Symbolic Logic, the chapter on Propositional Functions and many others, are quite obstruse. But the chapter on Infinity, the Infinitesimal, and Continuity, and many others are quite valuable to the ordinary student of mathematics.

The second volume, the author informs us, will be addressed exclusively to mathematicians. The whole work will constitute a valuable addition to the library of philosophers and mathematicians alike. B. F. F.

Plane and Solid Geometry, on the Suggestive Method, with Numerous Exercises and a Brief Course on Loci of Equations and on Conic Sections, by A. W. Williamson, Professor of Mathematics in Augustana College, Rock Island, Ill. 8 vo. Cloth. x+283 pp. Price, $1.00.

This work is based on the "heuristic method", and, while small in size, yet it contains more than the usual amount of matter offered in the common geometry course.

Many propositions are combined and easy exercises are given illustrating the important propositions. The book is a good one, and is worthy a cordial reception by teachers of Geometry who are seeking a good book on the subject and presenting it from the standpoint of the "heuristic method." B. F. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published Quarterly under the Auspices of Johns Hopkins University. Price, $5.00 per year, in advance

No. 3 of Vol. XXV contains the following articles: Isothermal-Conjugate Systems of Lines on Surfaces, by L. P. Eisenhart; Some Differential Equations Connected with Lines on Hypersurfaces, by G. O. James; On the Forms of Sextic Scrolls of Genus Greater than One, by Virgil Snyder; Geometry of the Cuspidal Cubic Cone, by Frederick C. Ferry.

The Annals of Mathematics. Editer by W. E. Byerly and others. Published Quarterly under the Auspices of Harvard University. Price, $2.00 per year, in advance.

No. 4 of Vol. IV, Second Series, contains the following articles: On the Character of Differential Equations, by E. R. Hedrick; On the Uniformity of the Convergence of Certain Absolutely Convergent Series, by Maxime Bocher; The Integral as the Limit of a Sum, and a Theorem of Dahamel's, by W. F. Osgood; On a General Relation of Continued Fractions, by R. E. Moretz; Note on the Equilateral Hyperbola, by J. A. Van Groos; A New Proof of the Generalized Wilson's Theorem, by G. A. Miller; A Sufficient Condition for the Maximum Number of Imaginary Roots of an Equation of the nth Degree, by E. B. Van Vleck.

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BIOGRAPHY.

JOHN DANIEL RUNKLE.

BY PROFESSOR H. W. TYLER, MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

John Daniel Runkle was born at Root, N. Y., October 11, 1822, and died at Southwest Harbor, Me., July 8, 1902, near the close of his eighthieth year.

The early years of life on the farm offered little opportunity for study, and he was already twenty-five when he entered the newly established Lawrence Scientific School of Harvard University. His name stands alone in the catalogue of 1848-49 as "student in mathematics." John W. Draper and James E. Oliver were fellow students; Josiah P. Cooke and William T. Harris, resident graduates. He was a member of the first graduating class, of 1851, with Joseph Le Conte and David A. Wells, receiving the degree of Bachelor of Science, and at the same time, for high scholarship, the honorary degree of Master of Arts.

The work of computation for the *Nautical Almanac* was carried on at this time in Cambridge by a staff including, among other men of subsequent eminence, Simon Newcomb, Asaph Hall, George W. Hill, T. H. Safford, and J. M. Van Vleck. Mr. Runkle's connection with the Almanac began in 1849, and continued in some form as late as 1884.

In 1852, he contributed, to the *Astronomical Journal*, papers on the "Elements of Thetis" and on the "Elements of Psyche."

In 1855, his "New tables for determining the values of coefficients, in the perturbative function of planetary motion, which depend upon the ratio of the

mean distances,'' were published as one of the Smithsonian Contributions to Knowledge.

In 1858, Mr. Runkle founded the *Mathematical Monthly*. Encouragement was received and formal indorsement given by the American Association for the Advancement of Science and by several educational bodies. The time for the publication of a long-lived mathematical journal was not, however, ripe, and only three volumes appeared. Following is a list of the more or less notable articles in the *Mathematical Monthly:*

Benjamin Peirce, Distribution of Points on a Line.

Rev. Thomas Hill, Derivations; Double Position; Lessons in Number.

George Eastwood, Mathematical Principles of Dealing.

W. P. G. Bartlett, Distribution of Points on a Line; Elements of Quaternions.

Truman H. Safford, Mathematical Theory of Music.

William Watson, Virtual Velocities; Descriptive Geometry of One Plane.

Chauncey Wright, Prismoidal Formula.

John B. Henck, Oval and Three Center Arches; Theorems of Pappus.

John Patterson, Relations between Minimum and Equilibrium; Process of Mathematical Development.

Arthur Cayley, Tangents to Conics.

Simon Newcomb, Theory of Probabilities.

Wm. Chauvenet, Great Circle on Mercator's Chart.

Rev. A. D. Wheeler, Indeterminate Analysis.

Mathew Collins, Centres of Similitude and Radical Axis.

From 1860 until his death, Professor Runkle's time and strength were almost continuously and exclusively devoted to the establishment and upbuilding of the Massachusetts Institute of Technology. He was first Secretary of the Institute, and at the opening of the school became professor of mathematics. In October, 1868, he became Acting President in consequence of President Rogers' serious illness, and in 1870 he was made President, holding the office for the following eight years.

The situation was a most exacting one, making altogether exceptional demands. The school, only five years old, was in no condition to lose the guidance of its founder. It had not yet gathered the momentum necessary for steady, straightforward progress. Opinions did and will differ as to President Runkle's judgment on the difficult question that, as time passed, pressed overwhelmingly upon him for solution. No man could have been more devotedly loyal to the school or to its founder, his predecessor and ultimately his successor. None could have shown more steadfast courage, not only against heavy odds, but too often with but feeble support.

The more notable events of the Runkle presidency were: The fruitless negotiations with Harvard University for a union; the establishment of the laboratories of mining engineering and metallurgy; the introduction of shop instruction and the foundation of the School of Mechanic Arts; the development of pro-

fessional summer schools in the field; the beginnings of an engineering laboratory; the increased efficiency of military instruction and the summer encampment at Philadelphia in 1876; the erection of a gymnasium, including a lunch room; the admission of women as students.

In 1878, Dr. Runkle resigned the presidency of the Institute and spent the following two years in Europe.

It had been President Runkle's merit to be the first to appreciate the American need of mechanic arts instruction based on principles already successfully applied in Russia. He was primarially interested in it as an invaluable addition to existing engineering courses, but he also saw clearly its great potential significance for general secondary education, and so far as possible, under pressure of other needs, demonstrated this by the inauguration of the School of Mechanic Arts, in which boys of high school age were offered a two years' course, including mathematics, English, French, history, mechanical and freehand drawing, and shop work. His visit to Europe enabled him to make a study of Continental schools of similar purpose; and the results of this study are embodied in a paper presented to the Society of Arts in April, 1881, on "Technical and Industrial Education Abroad," in an extended contribution to the Report of the Massachusetts Board of Education for 1880-81, and in a "Report on Industrial Education" in 1884. Others have taken a more directly prominent share in the introduction and extension of mechanic arts or manual training in primary and secondary schools, but the actual experiment initiated by him in Boston had in its time wide influence and imitation.

As a teacher of mathematics, Professor Runkle found his highest usefulness and most congenial vocation,—a vocation to be happily continued for not less than twenty-one years. His teaching was characterized by stimulating, luminous, unconventional exposition, by quick, incisive questioning, by warm personal interest in his students, and by a constant substratum of uplifting earnestness and dignity. None of his students could fail to acquire admiring affection; very few could withstand the incentive to work.

Professor Runkle was a man of much intellectual quickness and strength, of ardent, but in later years serene, temperament, of warm and generous affection, of cordial, unaffected courtesy, in all the relations of life a sincere and loyal gentleman. Throughout his early and middle life he was a pioneer, first in the struggle for his own education and that of his brothers, next in the establishment and continuance of a much needed, but, as it turned out, premature mathematical journal, then and for many years in the development of the Massachusetts Institute of Technology, and the introduction of education in the mechanic arts. In all these undertakings his insight and courage were invaluable. He made President Rogers' plans for the Institute his own. He held steadfastly to its fundamental ideals, and, taking account of his scanty resources, made remarkable progress toward their fulfillment. The main changes he initiated have been abundantly justified by time, and he lived to see their fulfillment.

He was elected a Fellow of the Academy on the 26th of May, 1857, and served one year (1877-78) as Councillor.

CONCERNING THE BIBLIOGRAPHY OF MATHEMATICS.*

By PROFESSOR J. W. A. YOUNG.

I. The Character of the Bibliographic Problem in Mathematics.

Mathematics is without doubt the most impersonal of all the branches of human thought. "Views" are not tolerated within its domain; "schools of thought" in the ordinary acceptance of the term are unknown. Conjectures, beliefs, moral certitudes, even, are not given a place among the truths of mathematics, though they are usually more or less distinct stages in the discovery of such truths. Only those statements are admitted to the rank of established mathematical truths, which are accepted by every normal mind to which they are properly presented.

Mathematics has been characterized as the only science which is not concerned about the truth or falsity of its data. The form of reasoning is: *If these premises are true, and if no other premises are taken into account, then such a conclusion follows.* The second *if* is very important and is a part of every mathematical inference.

A consequence of the severe standard thus set is that mathematics revises its results little. The statements of Euclid are as valid today as two thousand years ago The mathematician does not overturn the results of his predecessors; he *extends* them. He may find that the earlier work did not cover all the possibilities that have been seen later, but he does not on that account impugn the validity of what was done, so far as it went.

A good example is found in Euclid's postulate of parallels, viz., "*Given a point and a straight line, there is one (and only one) straight line passing through the point and parallel to the given straight line.*"† For many centuries this postulate was accepted as a logical necessity in Geometry, but during the 19th century‡ it was clearly seen that there was no reason why other assumptions should not be made, either that *no* parallel can be drawn through the given point, or that more than one can be drawn and a consistent Geometry developed. Thus arose the non-Euclidean Geometries. But the validity of the Euclidean Geometry was not at all called into question. The change made was that what was once *the* geometry became *a* geometry among several.

I have cited this as an illustration of the general statement: *The growth of mathematics has been an evolution without a revolution.* Mathematics has as yet had no Copernicus, and expects none.

All this has had a profound influence on mathematical literature. What is written is either a first presentation of a new result, or a representation of it

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*An address delivered before the Chicago Bibliographical Society, April 30, 1903.

†By parallel straight lines, we understand here, straight unterminated lines in the same plane, which do not intersect.

‡Simon asserts (*Jahresbericht d. deutschen Math. Ver.* 1890, page 39) that the earliest established recognition that the parallel axiom is not a logical necessity was made by Gauss about 1792, who influenced the Bolyais and Lobatschewsky.

for one of several purposes that need not be enumerated here. The great geniuses of mathematics have not headed opposed and contending schools, but in harmonious, supplementary activity have added new truths to the mathematical treasury. In mathematics, therefore, questions of Bibliography center not about men but about topics. We need not expect to find thousands of works of comment and controversy written in a few years about the theories of any mathematician, however great.

II. What Are the Mathematician's Bibliographic Needs?

It is evident from what has just been said that the mathematical investigator needs primarily *topical* bibliographies. To inform ourselves thoroughly as to what has already been done is an almost indispensable preliminary to any serious research. This is needed both to enable the investigator to avail himself of what has already been accomplished, and also to prevent duplication. In other branches the independent re-doing of nominally the same piece of work is likely to be accomplished in such different form and spirit that the value of the one work is diminished little by the existence of the other. In mathematics real duplication is quite possible and, in fact, has frequently occurred.

III. How the Need Is Met.

1. The first refuge is always the *Fortschritte der Mathematik*, issued annually, giving brief descriptions of all available mathematical publications of the year in question. (I shall refer to this journal hereafter simply as the *Fortschritte*). It covers the period back to 1870, and references to all the earlier publications of importance are to be expected in various connections in the publications of that period. These references often amount to a very fair bibliography of the topic to date. I shall mention other publications (made and projected), covering the literature since 1800, which is by far larger and more important part of the whole.

The chief disadvantage of the *Fortschritte* is that it is about three years behind time. (The volume for 1900, for example, has just been completed.) Several other publication bring us nearer to date.

2. The most important of these is the *International Catalogue of Scientific Literature*, published for the International Council by the Royal Society of London. The first annual issue (for 1901) was published November, 1902. It gives titles only, classified on a decimal system, four digits in each number. The issue contains two catalogues, subject and author, each given in four languages, English, French, German, Italian.*

3. The International Congress for Bibliography of Mathematical Science held in Paris, 1889, appointed a commission to secure a list of titles of works from 1800 to 1889, with decennial supplement thereafter. The work done has taken the form of card titles,—published in series of abont 1000 each,—and the publication is still in progress.

4. *The Revue Semestrielle des Publications Mathématiques*, published under the

*A fuller description of this important catalogue will follow in a later number of the Monthly.

auspices of Mathematical Society of Amsterdam, now in its twelfth volume, comes nearest to covering the literature to date. The part for April to October, 1892, for example, appeared in February, 1903. The journals only are covered, and a few words of abstract are sometimes given.

5. *The Bolletino di Bibliografia e storia delle scienze matematiche* (in its 6th volume) is given mainly to reviews of new books.

These are the chief periodicals given up to bibliographic work exclusively. Among them, the *Fortschritte*, the oldest, still remains the most valuable aid so far as it goes, on account of its abstracts which range from a few lines to several pages according to circumstances.

A comparison of numbers of titles may be of some interest. The Royal Society Catalogue for 1901 contains 1506 titles in pure mathematics (2096 in Astronomy). The initial volume is necessarily somewhat incomplete, as quite a complex organization of national bureaus, readers, etc., has to be got into running order. The *Fortschritte* for 1900 contains (estimate) about 2600 titles, including applied mathematics, geodesy, and astronomy.

6. A most serviceable aid to the working mathematician, especially on the bibliographic side, is *L'Intermédiaire des Mathématiciens* (monthly; founded in 1894). This journal is intended as a medium for interchange of information between working mathematicians. The form is that of questions and answers. The questions represent genuine needs of the writer arising in his researches; often they are directly bibliographical, while others relate to solution of specific problems. The answers are bibliographical when possible; that is, references which will put the questioner on the track of what he wants; though if a reader can give the derived solution himself, but knows of no previous publication of it, the solution itself will be published as reply if brief; if extended, it is published elsewhere, and reference made in *Intermédiare*. The *Intermédiaire* is now completing its first decade; it has met a cordial reception among mathematicians and has done good service in making generally available many pieces of information which would otherwise have remained unutilized.

7. Another phase of general mathematical bibliography is that of classified collections of results, with references to sources for proofs. Many such have been made of varying character and thoroughness, dealing with only a small field; others attempting to cover the entire range of mathematics. I can not now enter upon an enumeration of them; to prepare an exhaustive list would be a bibliographic task in itself. But I mention simply two: Carr, Synopsis of Pure Mathematics (London, 1886, pp. xxxvii, 935), and Hagen, Synopsis der Höheren Mathematik. The latter work by J. G. Hagen, Director of the Observatory, Georgetown College, Washington, D. C., is a stupendous undertaking, planned to be completed in four quarto volumes, of which two have appeared, and form most valuable works of reference.

8. But no one man can prepare the best possible compendium of all mathematics, or even of any large branch of it. Realizing this there is being prepared, under the auspices of the academies of science in Munich and Vienna, and

the Scientific Society of Göttingen, jointly, an *Encyclopedia of Mathematics* which is to be a compendium of the status of mathematical science today, with bibliographic references since the beginning of the nineteenth century. The work is planned in seven volumes, each in many parts, and over sixty prominent mathematicians are connected with the first three volumes, dealing with so-called pure mathematics (78 topics, each entrusted to a specialist). The work is now in process of appearing (first installment issued in 1898) and constitutes the best general survey of the field of mathematics which exists at present.*

9. Parallel with the encyclopedia, which is purely bibliographical, its publishers (Teubner, Leipzig) are also getting out a *series of mathematical text-books*, which will include proofs as well as results, and which are intended to give a fuller survey of the present state of the various topics taken up than can be given within the narrow limits of the encyclopedia. The authors are mathematicians of high rank. In a number of instances the writer of a section in the encyclopedia furnishes a corresponding treatise as text-book, and the series will thus constitute a most valuable supplement to the encyclopedia.*

10. The *Jahresbericht der Deutschen Mathematiker Vereinigung*, founded 1891, publishes excellent, sometimes elaborate reports on the development and on the present status of special topics, which reports often constitute most valuable bibliographies. Thus there have already been published reports on the Theory of Invariants, the Theory of Algebraic Functions, the Theory of Algebraic Number-fields, Synthetic Geometry, text-books on Infinitesimal Calculus, Theory of Probabilities, Point Manifolds, Kinetic Problems of Scientific Technology, Development According to Oscillatory Functions, and a number of others are announced for the near future.†

11. There has just been published (1903) a German work, *Mathematischer Bücherschatz*, by Wölffing (Teubner, Leipzig), which aims to give a complete list of all non-periodical advanced mathematical works of the world, published in the Nineteenth century. The titles are arranged under 313 heads, and alphabetically by authors under these heads.

12. The *Bibliotheca Mathematica* (Stockholm), though primarily devoted to the history of mathematics, occasionally contains good bibliographies of special topics.

IV. Classifications of Mathematics.

Prerequisite to any general bibliographic work like any of those men-

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*A fuller account of the encyclopedia and the series will be given in a subsequent number of the Monthly.

†Meyer: *Bericht ueber d. gegenwaertigen Stand d. Invarianten-theorie, 1890.*

Brill u. Noether: *Die Entwickelung d. Theorie d. algebraischen Functionen in aelterer u. neuerer Zeit, 1893.*

Hilbert: *Die Theorie d. algebraischen Zahlkoerper, 1895.*

E. Koetter: *Die Entwickelung d. synthetischen Geometrie, 1897.*

Bohlmann: *Uebersicht ueber die wichtigsten Lehrbuecher d. Infinitesimalrechnung von Euler bis auf die heutige Zeit, 1897.*

Czuber: *Die Entwickelung der Wahrscheinlich keitstheorie, 1898.*

Schoenflies: *Entwickelung d. Lehre d. Punktmannigfaltigkeiten, 1899.*

Heun: *Die Kinetische Probleme d. wissenschaftlischen Technik, 1900.*

Burkhardt: *Entwickelungen nach oscillirenden Funktionen, 1901.*

tioned, is some classification of mathematical topics. The *Encyclopedia* entitles its seven volumes as follows (thus forming a first classification):

I. Arithmetic and Algebra
II. Analysis
III. Geometry
— Pure Mathematics.

IV. Mechanics
V. Theoretic Physics
VI. Geodesy and Mathematical parts of Geophysics and Astronomy
VII. Philosophical, Historical, and Pedagogical Questions.
— Applied Mathematics.

Each volume is subdivided into twenty or more topics which are assigned to various writers, each one of whom organizes the matter within his own topic as may seem best to him.

A somewhat similar classification is continued in the *Fortschritte*, while those of the *Reportoire* and of the Royal Society Catalogue are each more elaborate. The former has five indices (example, L'3bα1), while the latter is on a decimal system with four digits (example, 3470).

The symplicity of the latter system produces a favorable first impression, but as it has been made public only so recently it is not yet possible to speak concerning its merits in comparison with the earlier schemes of classification, or to prognosticate how well it will be found adapted to the actual work of classification and reference.

V. Needs of Mathematical Bibliography.

The bibliographical helps which are periodical (appearing annually or more frequently) cover the whole field in each issue. Topical bibliographies are also desirable with sub-classifications and reference to minor portions of works on other topics, which deal with the topic in hand.

The *Fortschritte* makes little attempt at sub-classification. The Royal Society Catalogue has a more elaborate classification and some cross references—but in a comprehensive bibliography of all mathematics sub-classification cannot with advantage be carried to the same extent as in the bibliography of a special topic.

Bibliographies of the latter type may achieve two important ends, viz:

1. All the material in a single classification independently of the year of publication.

2. Greater sub-classification than is possible in a more general bibliography.

Topical bibliographies, whether of large subjects or of narrow, highly special fields are now perhaps the most important bibliographic disiderata in mathematics. Much work of this sort has already been done. It would be a bibliographic undertaking of considerable magnitude and importance to prepare a fairly complete list of such special bibliographies.*

*I mention simply a few by American writers:

At the International Congress of Mathematicians held at Paris in 1900, Mr. Ed. Maillet recommended the preparation and publication of bibliographic notices for the assistance of those who wish to take up the study of specific problems. He gives as a model:

"The last theorem of Fermat: $x^m+y^m \neq z^m$, $m>2$. Fermat announced without proof the theorem: The indeterminate equation $x^m+y^m=z^m$ cannot be satisfied by integers if $m>2$. This theorem has not yet been completely solved. To begin its study read:

Serret: *Algébra Supérieure;*
Legendre: *Mém. de l' Institute, 1823;*
Dedekind-Dirichlet: *Zahlentheorie;*
Bachmann: *Kreistheilung;*
(or, in place of the last two books, Bachmann: *Zahlentheorie*);
Kummer: *Jour. d. Math.* t. XVI, *Jour. f. Math.* 1837, 1846, 1847, 1850, *Abh. d. Wiss. z. Berlin*, 1857;
Mirimanoff: *Journal für Mathematik;*
Hilbert, Maillet: *Mém. de l' Assoc. franc. pour l' avanc des Sciences*, 1897; *Comptes rendus de l'Acad. des Sc.*, July, 1899, and *Acta. Mat.*, 1900.

The first attack might be restricted to an attempt to prove that $x^{\lambda t}+y^{\lambda t}=z^{\lambda t}$, λ a prime, is impossible in prime integers, for every t superior to a certain limit function of λ; the theorem is already established when x, y, z are prime to λ, and among themselves.

Sketches like the above for a great number of mathematical problems, whether difficult or not, outlining in a few lines the state of the question, and the problems to be solved, are very desirable; *Intermédiaire* stands ready to publish them."

On the same general line, Hilbert read at the Congress a very important and inspiring paper on the future problems of mathematics. This paper may be found in French, in the *Congress Reports*, 1900; in German, in the *Gött. Nachrichten*, 1900; also in the *Archiv. f. Math. u. Physik*, 1900; in English, in the *Bulletin of the American Mathematical Society*, 1902.

What precedes relates to the newer topics and fields, those which are the scene of present mathematical activity, and where a good bibliography may contribute directly to further investigation.

Bibliographies in the field of the older and settled questions are, however, also desirable, both because these questions stand in more or less close relation to unsettled questions, and also because a careful study of the genesis of any notion, its treatment and development to a completed form is always instructive and stimulating.

Halsted: *Bibliography of Non-Euclidean Geometry*, American Journal of Mathematics, 1879.
Miller: *On Recent Progress in the Theory of Groups of a Finite Order*, Bulletin American Mathematical Society, 1899; *of Infinite Order, ibid.*, 1900.
Dickson: *Report on the Recent Progress in the Theory of Linear Groups*, Bulletin American Mathematical Society, 1899.
The Known Systems of Simple Groups and their Isomorphisms, Report of the Paris Congress, 1900, page 225.
Easton: *Bibliograyhy of Substitution Groups*, published separately, Philadelphia, 1902.
Macfarlane: *Bibliography of Quaternions* (in preparation).

A DISCUSSION OF THE CASES WHEN TWO QUADRATIC EQUATIONS INVOLVING TWO VARIABLES CAN BE SOLVED BY THE METHOD OF QUADRATICS.*

By MISS ADELAIDE DENIS, Graduate Student, Colorado College.

1. The treatment of simultaneous quadratics in our elementary text-books is most unsatisfactory to the teacher. The author sometimes begins with the theorem, "The solution of a system of quadratic equations involving two variables in general requires the solution of a biquadratic." More often, no mention is made of the general theorem. Three cases are stated where special devices make possible the solution by quadratics. Then follows a set of problems, some under these three heads, many not. The pupil is left to use his ingenuity, with more or less suggestion from others, in solving them.

Chrystal says: "A moderate amount of practice in solving puzzles of this description is useful as a means of cultivating manipulative skill, but he (the student) should beware of wasting his time over what is, after all, merely a chapter of accidents."

The purpose of the following paper is to attempt to remove the problem from the category of a "chapter of accidents," or the realm of puzzles, and make it possible for the teacher, if not the pupil, to determine when the solution of such a system is possible by the method of quadratics. The conditions that the given equations can be solved entirely by the method of quadratics are obtained. In certain cases where these tests fail, it is still possible to obtainq uadratic factors, or a linear factor which will give a partial solution. The latter case is of especial interest, as indicated by the discussions in Vol. VI, pages 13-14, and Vol. VII, page 169, of THE AMERICAN MATHEMATICAL MONTHLY.

For the following work, Chrystal's *Text-book of Algebra*, Third Edition, Part I, pages 416-417; Burnside and Panton's *Theory of Equations*, Third Edition, pages 129-130; and Dr. K. L. Bauer's article in *Hoffmann's Zeitschrift*, 1874, page 317, have been found most helpful.

2. Let the given system of equations be represented by

$$ax^2+by^2+cxy+dx+ey+f=0.....(\mathrm{I})$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0.....(\mathrm{II}).$$

Following Chrystal's suggestion as to method of solution, let

$$cy+d=p, \quad by^2+ey+f=q,$$

$$c'y+d'=p', \quad b'y^2+e'y+f'=q'.$$

Substituting these values in (I) and (II), we get

* Submitted through Professor Cajori and somewhat condensed by Editor Dickson.

$$ax^2+px+q=0, \quad a'x^2+p'x+q'=0.$$

Eliminating x^2, and finally, also x itself, we get

$$(aq'-a'q)^2-(ap'-a'p)(pq'-p'q)=0.$$

Substituting the values assumed for p and q,

$$[a(b'y^2+e'y+f')-a'(by^2+ey+f)]^2-\{[a(cy'+d')$$
$$-a'(cy+d)][(cy+d)(b'y^2+e'y+f')-(c'y+d')(by^2+ey+f)]\}=0\text{.....(III)}.$$

In expanding the last form the coefficients are such that determinants can be used to advantage. The determinant $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ or $ab'-a'b$ will for convenience be written in the form $[ab']$. The equation takes the form

$$Ay^4+By^3+Cy^2+Dy+E=0\text{.....}(Q).$$

where

$A=[ab']^2+[ac'][bc']$,

$B=2[ae'][ab']-[ac'][ce']+[ac'][bd']+[ad'][bc']$,

$C=[ae']^2+2[ab'][af']-[ac'][cf']-[ac'][de']-[ad'][ce']+[ad'][bd']$,

$D=2[ae'][af']-[ac'][df']-[ad'][cf']-[ad'][de']$,

$E=[af']^2-[ad'][df']$.

3. If the quartic (Q) is separable into quadratic factors, the quartic may be said to be irreducible or reducible, according as the coefficients do or do not involve irrational numbers.

I. Irreducible quartic:

(1) $[(r+\sqrt{m})y^2+ny+p][(r-\sqrt{m})y^2+ny+p]=0.$

(2) $[my^2+(r+\sqrt{n})y+p][my^2+(r-\sqrt{n})y+p]=0.$

(3) $[my^2+ny+(r+\sqrt{p})][my^2+ny+(r-\sqrt{p})]=0.$

II. Reducible quartic:

(1) Solvable by method of quadratics. The same forms as I, if $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$ are rational.

(2) Solvable by factoring of quartic, but not by the method of quadratics, pure and simple:

(a) $(my^2+ny+p)(m'y^2+n'y+p')=0.$

(b) $(my^3+ny+py+q)(m'y+q')=0.$

4. All equations of the first form I, (1), can be written

$$(ry^2+ny+p)^2-my^4=0, \text{ or } (Xy^2+\frac{D'}{2}y+1)^2-Yy^4=0.....(IV),$$

in which the absolute term is obtained by dividing both factors by p; $D'=D/E$, D and E as in (Q), X and Y to be found.

Expanding (IV) and comparing the coefficients with the coefficients of the same powers in (Q), after dividing (Q) by E,

$$A/E=X^2-Y, \quad D'X=B/E, \quad D'/4+2X=C/E.$$

Equating the values of X derived from the last two,

$$D^3+8BE^2=4CDE.....(J).$$

Therefore (J) is the condition that the quartic be capable of being resolved into factors of the first form.

5. All equations of the second form I, (2), can be written

$$(my^2+ry+p)^2+ny^2=0 \text{ or } [\sqrt{(a)}y^2+Xy+\sqrt{E})-Yy^2=0.....(V),$$

in which A and E are the same as in (Q), X and Y to be found. Expanding and comparing the coefficients with the coefficients of the same powers in (Q),

$$2\sqrt{a}\,X=B, \quad X^2+2\sqrt{(AE)}-Y=C, \quad 2X\sqrt{E}=D.$$

$\therefore B^2E=D^2A.....(K)$.

Therefore (K) is the condition that the quartic (Q) shall be capable of being resolved into factors of the second form.

6. All equations of the third form can be written

$$(my^2+ny+r)^2-p^2=0 \text{ or } (y^2+\frac{B'}{2}y+X)^2-Y=0.....(VI),$$

in which $B'=B/A$, B and A as in (Q), X and Y to be found. Expanding and comparing coefficients with the coefficients of the like powers in (Q), after dividing by A,

$$2X+B'^2/4=C/A, \quad B'X=D/A, \quad X^2-Y=E/A.$$

$\therefore B^3+8A^2D=4ABC.....(L)$.

Therefore (L) is the condition that the quartic (Q) be capable of being resolved into factors of the third form.

The test (L) is the one obtained by Dr. K. L. Bauer in his article referred to in the introduction.

As an example under the case I of irreducible quartic, take

$$5y^2+2y-x-3=0, \quad x^2+6x+40y^2+16y+20=0.$$

Then (Q) becomes

$$25y^4+20y^3+44y^2+16y+11=0.$$

Trying successively the tests (J), (K), (L), the last one gives the identity

$$(20)^3+8.(25)^2.16=4.25.20.44.$$

Then $X=D/B=\frac{4}{5}$, $Y=X^2-E/A=\frac{1}{5}$.

$$[y^2+\tfrac{2}{5}y+\tfrac{4}{5}+\sqrt{\tfrac{1}{5}}][y^2+\tfrac{2}{5}y+\tfrac{4}{5}-\sqrt{\tfrac{1}{5}}]=0.$$

7. If (Q) is separable into rational factors, these factors may be both quadratic, or one linear and one cubic. If the factors are quadratic, the roots of the quartic may be all irrational, two irrational and two rational, or all rational. The general form for such a quartic is,

$$(my^2+ny+p)(m'y^2+n'y+p')=0 \ldots\ldots (\text{VII}).$$

If the radical terms disappear in the three forms already considered, the resulting forms are special cases under (VII). If, therefore, the tests (J), (K), or (L) give a result in which $\sqrt{Y}$ is rational, the roots of the quartic may be obtained by the method used for the irreducible case.

8. Consider next the special devices that lead to the factoring of quartic, when the previous direct tests fail and the quartic cannot be solved by the method of quadratics.

If the quartic (Q) is separable into two quadratic factors, the general form (VII) has been given. One of these factors may be obtained as follows. Dividing (Q) by A, the coefficient of y^4, and transforming the resulting equation so as to remove fractional coefficients, if this be necessary, let the resulting quartic be

$$y^4+a_1y^3+a_2y^2+a_3y+a_4=0 \ldots\ldots (Q').$$

If (Q') is separable into quadratic factors, let

$$y^2+\alpha y+\beta \ldots\ldots (P)$$

be one of those factors. Dividing the first member of (Q') by this factor, the remainder is

$$[(a_3-a_2\beta+\alpha\beta)-(a_2\alpha-\alpha\beta-a_1\alpha^2+\alpha^3)]y+a_4-(a_2\beta-\beta^2-a_1\alpha\beta+\alpha^2\beta)\text{.....}(R),$$

and the quotient is,

$$y^2+(a_1-\alpha)y+(a_2-\beta-a_1\alpha+\alpha^2)\text{.....}(P').$$

If $(R)=0$, then (P) and (P') are the quadratic factors of (Q').

The two conditions that $(R)=0$, identically, are

$$(1)\ \ a_3-a_1\beta+\alpha\beta=a_2\alpha-a_1\alpha^2+\alpha^3-\alpha\beta,$$

$$(2)\ \ a_4=a_2\beta-\beta^2-a_1\alpha\beta+\alpha^2\beta.$$

From them

$$\frac{a_3-a_1\beta+2\alpha\beta}{\alpha}=a_2-a_1\alpha+\alpha_2,\quad \frac{a_4}{\beta}+\beta=a_2-a_1\alpha+\alpha_2.$$

Equating the first members, and solving for α,

$$\alpha=(a_3\beta-a_1\beta^2)/(a_4-\beta^2).$$

β must be an integral factor of a_4. Therefore if a value for β can be found by factoring a_4, which will give the above expression for α integral, the factor (P) can be determined and second factor (P') readily found.

If the quartic is reducible, and one factor linear and one an irreducible cubic, the solution cannot be effected by the method of quadratics.

If however, (Q) be divided by A, and the resulting equation transformed to remove fractional coefficients, if necessary, the single real and rational root may be found by the Remainder Theorem. Thus one set of values for x and y in the original equations may be found.

There are some special cases under this general one that are worthy of consideration. Systems of quadratics of the form

$$x^2+y=a,\quad y^2+x=b,$$

have been of especial interest. (See end of §1).

It can readily be shown, by applying the tests (J), (K), and (L), that the resulting quartic does not come under any of the cases solvable directly by the method of quadratics. But sometimes by a special device we can find one set of values for x and y. This is the case when the quartic can be separated into one linear and one irreducible cubic factor. For instance, let the general form of such a system be

$$\left.\begin{matrix}(ax+by+c)^2-d^2=e-(fx+gy+h)\\(fx+gy+h)^2-e^2=d-(ax+by+c)\end{matrix}\right\}\text{.....}(F).$$

That $$ax+by+c=d,\quad fx+gy+h=e,$$

can be seen by inpection. If the given equations can be arranged as in (F), the solution as far as obtainable by this device, is readily completed.

To separate (I) and (II) as given in (F), extract the square roots of the first members of (I) and (II) as far as possible. If the remainder obtained in the first is the root obtained in the second, and the remainder in the second the root in the first, the equations may be written in the form (F).

A special case of the above general form is

$$x^2-d^2=e-y, \quad y^2-e^2=d-x,$$

in which $x=d$, $y=e$. See the references at end of §1.

Another of these special cases is the one in which the single real and rational root is a quartic can be obtained by the method next explained. The following two theorems will be used, taken from Eugen Netto's *Vorlesungen über Algebra*, Vol. I, page 56:

(1) If all the coefficients, c_1, $c_2 .....c_n$ of the polynomial

$$f(y)=y^n+c_1y^{n-1}+.....+c_{n-1}y+c_n$$

are integral and divisible, without a remainer, by a prime number p, but c_n is not divisible by a higher power of p than the first, then $f(y)$ is irreducible.

(2) If all the coefficients, c_1, $c_2 .....c_n$ of the polynomial

$$f(y)=y^n+c_1y^{n-1}+.....+c_{n-1}y+c_n$$

are divisible by a prime number p, but c_{n-1} is not divisible by a higher power of p than the first, $f(y)$ is either irreducible or is reducible to a factor of the first degree and an irreducible factor of the $(n-1)$th degree.

If the coefficients a_1, a_2, a_3, a_4 of the quartic (Q') are divisible by a prime number p, but a_3 is not divisible by a higher power of p than the first, and the quartic is reducible to a linear factor and an irreducible quartic, one root of the quartic, and one set of values of x and y can be ascertained without recourse to the regular algebraic solution. Let the cubic be

$$y^3+c_1y^2+c_2y+c_3=0, \quad c_1=p^nr, \quad c_2=p^mr, \quad c_3=pr_2.$$

If the factor $y+a$ is introduced,

$$y^4+(a+p^nr)y^3+(ap^nr+p^mr_1)y^2+(ap^mr_1+pr_2)y+apr_2=0.....(M).$$

Let $a+p^nr=B, \quad ap^n+p^mr_1=C, \quad ap^mr_1+pr_2=D, \quad apr_2=E.$

Then a system of quadratic equations giving the quartic (M) is

$$2y^2+By=x, \quad x^2+(4C-B^2)y^2+4Dy+4E=0.$$

9. There are certain forms of simultaneous quadratics which are readily recognized as solvable by the method of quadratics. It may be interesting to place these according to the preceding discussion. The classification given by Chrystal, Algebra, Part I, page 417, has been followed quite closely.

I. Two roots zero. The quartic (Q) assumes the form

$$Ay^4+By^3+Cy^2=0 \text{ or } (Ay^2+By+C)y^2=0.$$

It is evident that the conditions $D=0$, $E=0$ are satisfied if

$$ad'=a'd,\ \ af'=a'f,\ \ df'=d'f,\ \text{ or }\ a/a'=d/d'=f/f' \ldots\ldots (N).$$

But if $D=0$, $E=0$ tests (J) and (K) are satisfied. Therefore the system of quadratics is solvable by the method of quadratics.

II. Two roots infinity. The quartic (Q) assumes the form

$$Cy^2+Dy+E=0.$$

It is evident that a sufficient condition for the vanishing of A and B is

$$ab'=a'b,\quad ac'=a'c,\quad bc'=b'c,\ \text{ or }\ a/a'=b/b'=c/c'.$$

But if $A=0$, $B=0$, the tests (K) and (L) are satisfied; therefore the quadratics are solvable as before.

III. The quartic contains only the even powers of y. The quartic (Q) assumes the form $ay^4+cy^2+E=0$. A sufficient condition for the vanishing of B and D is

$$d=0,\quad d'=0,\quad e=0,\quad e'=0.$$

But these are the conditions that (I) and (II) are homogeneous, and the tests (J), (K), and (L) are satisfied, so that two homogeneous equations and any other systems producing a quartic (Q) containing only the even powers of y, are solvable by the method of quadratics.

IV. If the quartic is a reciprocal equation of the form

$$(1)\ \ Ay^4+By^3+Cy^2+By+A=0,$$

then $A=E$, $B=D$. But if $A=E$,

$$\{ab'\}^2+\{ac'\}\{bc'\}=[\{af'\}^2-\{ad'\}\{df'\}],$$

a condition which is satisfied if $b=f$, $b'=f'$, $c=d$, $c'=d'$. If these conditions hold, $B=D$, so that the quartic is reciprocal.

But if $A=E$, $B=D$, the test (K) is satisfied. Therefore a system of quadratics producing a reciprocal quartic of the form (1) is solvable as before.

If the quartic assumes the reciprocal form

$$(2)\ Ay^4+By^3-By-A=0,$$

the above test (K) does not hold. But if the equation be divided by A, it is seen as in §8 that $\alpha=0$, so that y^2-1 is a factor of the reciprocal equation. The remaining factor is readily found.

V. If the quadratic equations are symmetrical,

$$a=b,\quad d=e,\quad a'=b',\quad d'=e'.$$

Equations (I) and (II) then become, after division by a and a',

$$x^2+y^2+c_2xy+d_1x+d_1y+f_1=0 \ldots\ldots (\mathrm{I}'),$$

$$x^2+y^2+c_1xy+d_2x+d_2y+f_2=0 \ldots\ldots (\mathrm{II}').$$

Subtracting, $$c_3xy+d_3x+d_3y+f_3=0 \ldots\ldots (\mathrm{III}').$$

Substituting $x=\mu+\nu$, $y=\mu-\nu$ in (I′) and (III′),

$$a_1\mu^2+b_1\nu^2+2d_1\mu+f_1=0 \ldots\ldots (\mathrm{IV}'),$$

$$c_3\mu^2-c_3\nu^2+2d_3\mu+f_3=0 \ldots\ldots (\mathrm{V}').$$

From (IV′) and (V′), ν^2 may be eliminated, and the resulting quadratic in μ readily solved. Or the values for a, a', etc., may be substituted in (III). In this case the terms in ν^3 and ν will vanish and the resulting quartic in ν comes under the special case III just discussed.

AN EXTENSION TO CENTRAL CONICOIDS OF A THEOREM CONCERNING THE SEGMENT OF A SPHERE.

By G. W. GREENWOOD, McKendree College.

Consider the sphere, the cylinder, and the cone whose equations are, respectively,

$$x^2+y^2+z^2=1 \ldots\ldots (1),$$

$$x^2+y^2=1 \ldots\ldots (2),$$

$$x^2+y^2=z^2 \ldots\ldots (3).$$

It is shown in text books that the volume of (1) included between planes parallel to the plane of x, y is equal to the volume of the segment of (2) diminished by the that of the segment of (3) included between the same planes.

If we employ the equations

$$x^2/a^2+y^2/b^2+z^2/c^2=1.....(4),$$

$$x^2/a^2+y^2/b^2=1.....(5),$$

$$x^2/a^2+y^2/b^2=z^2/c^2.....(6),$$

for (1), (2), (3), respectively, the theorem is still true. If for (4) we substitute either of the equations

$$x^2/a^2+y^2/b^2=1+z^2/c^2.....(7),$$

$$x^2/a^2+y^2/b^2=z^2/c^2-1.....(8),$$

we get the theorems that the volume of a segment of (7) made by planes parallel to that of x, y is equal to the sum of the volumes of the corresponding segments of (5) and (6); and that of a segment of (7) is the volume of the corresponding segment of (6) diminished by that of the segment of (5).

The volume of a segment of (1) made by planes parallel to that of x, y is equal to the sum of the volumes of two cylinders whose bases are the bases of the segment and altitudes half that of the segment, together with the volume of a sphere to which the bases of the segment are tangent.

The corresponding theorems are as follows: The volume of a segment of (4) made by planes parallel to that of x, y is equal to the sum of the volumes of two cylinders whose bases are the bases of the segment and whose altitudes are half that of the segment, together with the volume of an ellipsoid to which the bases of the segment are tangent, similar to (4) and similarly placed.

Furthermore, the volume of a segment of (7) or (8) made by planes parallel to that of x, y is equal to the volumes of two cylinders whose bases are the bases of the segment, diminished by the volume of an ellipsoid to which the bases are tangent, whose axes are proportional to those of (7) or (8) and respectively parallel to them.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

178. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A_n being the arithmetic mean of the nth powers of the numbers less than p and prime to it, find a relation between A_3, A_2 and p.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The number n of positive integers less than p and prime to it is

$$n=p(1-1/a)(1-1/b)(1-1/c)(1-1/d).....$$

The sum of the squares of all such numbers is

$$S_2=\tfrac{1}{3}p^3(1-1/a)(1-1/b)(1-1/c)(1-1/d)\,.....+\tfrac{1}{6}p(1-a)(1-b)(1-c)(1-d)\,.....$$

and the sum of the cubes is

$$S_3=\tfrac{1}{4}p^4(1-1/a)(1-1/b)(1-1/c)(1-1/d).....+\tfrac{1}{4}p^2(1-a)(1-b)(1-c)(1-d).....$$

$$\therefore A_2=S_2/n=\tfrac{1}{3}p^2+\tfrac{1}{6}\cdot\frac{(1-a)(1-b)(1-c)(1-d).....}{(1-1/a)(1-1/b)(1-1/c)(1-1/d).....}.$$

$$=\tfrac{1}{3}p^2+\tfrac{1}{6}B,\text{ suppose.}$$

$$A_3=S_3/n=\tfrac{1}{4}p^3+\tfrac{1}{4}pB.$$

$$\therefore B=6A_2-2p^2=\frac{4A_3-p^3}{p}.$$

$$\therefore 6A_2p-2p^3=4A_3-p^3,\text{ or } 6A_2p-4A_3=p^3,\text{ or } p^3-6A_2p+4A_3=0.$$

179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left(\frac{q^2}{p}-\frac{p^3}{5q}\right)=0.$$

No solution of this problem has been received.

180. Proposed by the late JOSIAH H. DRUMMOND.

If r/s is such a value of p as makes $m/(p^2-2)$ integral, prove that $(3r+4s)/(2r+3s)$ is another such value, so that an indefinite number of integral values may be obtained.

Also, if r/s is such a value of p as makes $2m/(p^2-2)$ integral, prove that $2(r+s)/(r+2s)$ is also such a value.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

When $p=r/s$, $m/(p^2-2)=ms^2/(r^2-2s^2)$.....(1).

When $p=(3r+4s)/(2r+3s)$, $m/(p^2-2)=m(2r+3s)^2/(r^2-2s^2)$.....(2).

Since (1) is integral, (2) is also, for we can take $m=n(r^2+2s^2)$.

When $p=r/s$, $2m/(p^2-2)=2ms^2/(r^2-2s^2)$.....(3).

When $p=2(r+s)/(r+2s)$, $2m/(p^2-2)=m(r+2s)^2(r^2-2s^2)$.....(4).

Since (3) is integral, (4) is also.

Also solved by the *PROPOSER*.

GEOMETRY.

200. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of eight points of contact of the four common tangents of two concentric coaxial ellipses.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/a^2+y^2/b^2=1$, and $x^2/c^2+y^2/d^2=1$, be the equations to the ellipses, $c>a$, $b>d$.

Since $xx_1/a^2+yy_1/b^2=1$, and $xx_2/c^2+yy_2/d^2=1$, are two equations for the same line, we get $x_2=c^2x_1/a^2$, $y_2=d^2y_1/b^2$.

$\therefore x_1^2/a^2+y_1^2/b^2=1$, and $c^2x_1^2/a^4+d^2y_1^2/b^4=1$, give

$$x_1=\frac{a^2\sqrt{[b^2-d^2]}}{\sqrt{[b^2c^2-a^2d^2]}}=a^2m \text{ (suppose)},\ y_1=\frac{b^2\sqrt{[c^2-a^2]}}{\sqrt{[b^2c^2-a^2d^2]}}=b^2n,$$

$$x_2=c^2m,\ y_2=d^2n.$$

Hence $mx\pm ny\pm1=0$ represents the four common tangents. While $(c^2m,\ d^2n)$; $(a^2m,\ b^2n)$; $(-a^2m,\ b^2n)$; $(-c^2m,\ d^2n)$; $(-c^2m,\ -d^2n)$; $(a^2m,\ -b^2n)$; $(c^2m,\ -d^2n)$ are the eight points of contact. These eight points are situated on an ellipse, which is the locus required.

Let $x^2/A^2+y^2/B^2=1$ be this ellipse. Then $c^2m/A^2+d^4n^2/B^2=1$, also $a^4m^2/A^2+b^4n^2/B^2=1$.

$$\therefore A^2=\frac{(b^4c^4-a^4d^4)m^2}{b^4-d^4}=\frac{b^2c^2+a^2d^2}{b^2+d^2},\ B^2=\frac{(b^4c^4-a^4d^4)n^2}{c^4-a^4}=\frac{b^2c^2+a^2d^2}{a^2+c^2}.$$

$\therefore (b^2+d^2)x^2+(a^2+c^2)y^2=b^2c^2+a^2d^2$ is the locus.

201. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

Two plane sections of a right circular cone have their major axes AA', aa' coplanar, and Aa on one generator equal to $A'a'$ on the other. The projections of the sections on any plane perpendicular to the axis are confocal.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

Let $FG=ED=r$, $FD=2A$, $GE=2a$, $\angle EGK=\phi$, $\angle DFH=\theta$, $\angle GLF=x$, $\angle CGK=\angle CFH=\delta$, $CN=c$, $CM=d$.

Then $GF : LF = \sin x : \sin LGF$, or $r : LF = \sin x : \sin(\delta-\phi)$.

Also $ED : LD = \sin x : \sin LED$, or $r : 2A+LF = \sin x : \sin(\delta+\phi)$.

$\therefore LF\sin x = r\sin(\delta-\phi) = r\sin(\delta+\phi) - 2A\sin x$.

$\therefore A\sin x = r\cos\delta \sin\phi$.

Similarly, $a\sin x = r\cos\delta \sin\theta$. $\therefore A\sin\theta = a\sin\phi$.

$GN+EN=2a=2c\cos\delta \sin\delta \cos\phi/(\cos^2\phi-\cos^2\delta)$.

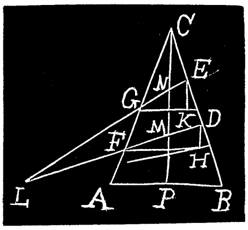
$\therefore c=a(\cos^2\phi-\cos^2\delta)/\cos\delta \sin\delta \cos\phi$,

$d=A(\cos^2\theta-\cos^2\delta)/\cos\delta \sin\delta \cos\theta$.

The axes of the projections on any plane perpendicular to CP are $A\cos\theta$ and B, and $a\cos\phi$ and b, respectively.

$$\left.\begin{array}{l} x^2\cos^2\phi(\tan^2\delta-\tan^2\phi)+y^2\tan^2\delta+2cx\sin\phi=c^2 \\ x^2\cos^2\theta(\tan^2\delta-\tan^2\theta)+y^2\tan^2\delta+2dx\sin\theta=d^2 \end{array}\right\} \text{are the equations to the ellipses.}$$

$$\therefore A^2=\frac{d^2\cos^2\theta \sin^2\delta \cos^2\delta}{(\cos^2\theta-\cos^2\delta)^2},\quad B^2=\frac{d^2\cos^2\theta \cos^2\delta}{\cos^2\theta-\cos^2\delta}.$$



$$a^2=\frac{c^2\cos^2\phi \sin^2\delta \cos^2\delta}{(\cos^2\phi-\cos^2\delta)^2}=b^2=\frac{c^2\cos^2\phi \cos^2\delta}{\cos^2\phi-\cos^2\delta}.$$

Substituting values of d and c,

$$A^2\cos^2\theta-B^2=\frac{d^2\cos^2\theta \sin^2\theta \cos^4\delta}{(\cos^2\theta-\cos^2\delta)^2}=A^2\sin^2\theta \cot^2\delta.$$

$$a^2\cos^2\phi-b^2=\frac{c^2\cos^2\phi \sin^2\phi \cos^4\delta}{(\cos^2\phi-\cos^2\delta)^2}=a^2\sin^2\phi \cot^2\delta=A^2\sin^2\theta \cot^2\delta.$$

$\therefore A^2\cos^2\theta-B^2=a^2\cos^2\phi-b^2$, and the sections are confocal.

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The equations $\sqrt{[l\alpha]}+\sqrt{[m\beta]}+\sqrt{[n\gamma]}=0$ and $l\beta\gamma+m\alpha\gamma+n\beta\alpha=0$ represent ellipses. If a, b, c are the sides of the triangle of reference, transform to Cartesian coördinates and find area of each ellipse.

Solution by the PROPOSER.

$\sqrt{[l\alpha]}+\sqrt{[m\beta]}+\sqrt{[n\gamma]}=0$, or

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\alpha\gamma-2ln\alpha\beta=0.....(1),$$

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=0.....(2).$$

Let ABC be the triangle of reference, B the origin, BC, BA the axes of coördinates. Then $\alpha=y\sin B$, $\gamma=x\sin B$, also $a\alpha+b\beta+c\gamma=ac\sin B$.

$\therefore \beta=[ac\sin B-a\alpha-c\gamma]b=\sin B[ac-ay-cx]/b$.

These values of α, β, γ in (1) and (2) give

$$[bn+cm]^2x^2+[am+bl]^2y^2+2[acm^2+abmn+bclm-b^2nl]xy$$

$$-2[ac^2m^2+abcmn]x-2[a^2cm^2-abclm]y+a^2c^2m^2=0\ldots\ldots(3).$$

$$clx^2+any^2+[al+cn-bm]xy-aclx-acny=0\ldots\ldots(4).$$

(3) reduces to $\{[bn+cm]x+[am+bl]y-acm\}^2-4b^2nlxy=0$.

If $A'x^2+B'y^2+2H'xy+2G'x+2F'y+C'=0$ be the equation (general) to an ellipse and $\triangle$ its discriminant, then area$=\dfrac{\pi\triangle\sin B}{[A'B'-H'^2]^{\frac{3}{2}}}$ where the axes are inclined at an angle B.

For (3), $\triangle=-4a^2b^4c^2l^2m^2n^2$, $A'B'-H'^2=4b^2lm[abn+acm+bcl]$.

$$\therefore \text{Area}=\tfrac{1}{2}\pi a^2bc^2\sin B\sqrt{\frac{lmn}{[abn+acm+bcl]^3}}.$$

If $l=\cos^2\frac{1}{2}A$, $m=\cos^2\frac{1}{2}B$, $n=\cos^2\frac{1}{2}C$, the ellipse becomes the in-circle.

$$\text{Area}=\frac{\pi a^2bc^2\cos\frac{1}{2}A\ \cos\frac{1}{2}B\ \cos\frac{1}{2}C\ \sin B}{2\sqrt{\{[bc\cos^2\frac{1}{2}A+ac\cos^2\frac{1}{2}B+ab\cos^2\frac{1}{2}C]^3\}}}$$

$$=\frac{\pi acs\sqrt{\{s[s-a][s-b][s-c]\}}\sin B}{2\sqrt{(\{[s-a]+s[s-b]+s[s-c]\}^3)}},\text{ where } s=\tfrac{1}{2}[a+b+c],$$

$$=\tfrac{1}{2}\pi ac\sin B\sqrt{\{s[s-a][s-b][s-c]\}}/s^2$$

$$=\pi\left(\frac{\text{area of triangle}}{s}\right)^2=\pi r^2.$$

For (4), $\triangle=-\frac{1}{2}a^2bc^2lmn$, $A'B'-H'^2=\frac{1}{4}\{4acln-[al+cn-bm]^2\}$.

$$\text{Area}=\frac{2\pi a^2bc^2lmn\sin B}{\sqrt{([\{4acln-[al+cn-bm]^2\}^3)}}.$$

Let $l=\sin A$, $m=\sin B$, $n=\sin C$, and the ellipse becomes the circum-circle.

$$\text{Area}=\frac{2\pi a^2bc^2\sin A\ \sin^2B\ \sin C}{\{4ac\sin A\sin C-[a\sin A+c\sin C-b\sin B]\}^{\frac{3}{2}}}$$

$$=\frac{2\pi a^2b^3c^3\sin A}{\sqrt{(\{4a^2c^2-[a^2+c^2-b^2]^2\}^3)}}=\frac{\pi a^2b^3c^3\sin A}{32\sqrt{(\{s[s-a][s-b][s-c]\}^3)}}$$

$$=\frac{\pi a^2b^2c^2}{[4 \text{ area of triangle}]^2}=\pi R^2.$$

CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College. Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly towards the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

Note by J. E. SANDERS, Hackney, Ohio.

The latter part of the solution given in the April MONTHLY is not correct. The dog cannot run towards the fox at all times and keep between the fox and the vertex of the hill. If we suppose only that he runs so that he is directly between the fox and the vertex, the following is the solution:

Let the surface of the hill be rolled out flat. Then the radius of the fox's path$=100\sqrt{2}$ feet$=a$. Put $n=\frac{5}{6}$, s=length of dog's path.

$$ds=[a/n]d\theta=\sqrt{\{r^2+[dr/d\theta]^2\}}.$$

$\therefore d\theta=\dfrac{ndr}{\sqrt{[a^2-n^2r^2]}}$, which gives $\theta=\sin^{-1}[nr/a]$ when integrated. When the dog catches the fox $r=a$, and

$\therefore \theta=\sin^{-1}n$ and $s=[a/n]\theta=120\sqrt{2}\sin^{-1}[\frac{5}{6}]=167.178$ feet.

No solution of Problem 165 has been received. See April number, 1903, page 115.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $[y^2+x^2]=a^2[x^2-y^2]$ round the axis of x.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$[y^2+x^2]^2=a^2[x^2-y^2]$.

$\therefore y^4+2x^2y^2+x^4=a^2x^2-a^2y^2$. $\quad\therefore y^2=\frac{1}{2}\{a\sqrt{[a^2+8x^2]}-a^2-2x^2\}$

$$\text{Volume}=2\int_0^a \pi y^2 dx=\pi\int_0^a \{a\sqrt{[a^2+8x^2]}-a^2-2x^2\}dx$$

$$=\pi\left[\frac{ax}{2}\sqrt{[a^2+8x^2]}+\frac{a^3}{4\sqrt{2}}\log\{\sqrt{[a^2+8x^2]}+2\sqrt{2}\,x\}-a^2x-\tfrac{2}{3}x^3\right]_0^a$$

$$=\frac{\pi a^3}{2}\left[\frac{1}{2\sqrt{2}}\log[3+\tfrac{1}{2}\sqrt{2}]-\tfrac{1}{3}\right]=\frac{\pi a^3}{2}\left[\frac{1}{2\sqrt{2}}\log[1+\sqrt{2}]^2-\tfrac{1}{3}\right]$$

$$=\frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\log[1+\sqrt{2}]-\tfrac{1}{3}\right].$$

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Integrate, $\displaystyle\int_0^a\int_0^b\int_0^c \frac{zdxdydz}{(x^2+y^2+z^2)^{\frac{5}{2}}}$.

Solution by the PROPOSER.

$$\int_0^a\int_0^b\int_0^c \frac{zdxdydz}{\sqrt{\{(x^2+y^2+z^2)^5\}}}=\tfrac{1}{3}\int_0^a\int_0^b\left[\frac{1}{\sqrt{\{(x^2+y^2)^3\}}}\right.$$

$$\left.\frac{1}{\sqrt{\{(c^2+x^2+y^2)^3\}}}\right]dxdy$$

$$=\frac{b}{3}\int_0^a\left[\frac{1}{x^2\sqrt{(b^2+x^2)}}-\frac{1}{(c^2+x^2)\sqrt{(b^2+c^2+x^2)}}\right]dx$$

$$=-\tfrac{1}{3}\left[\frac{\sqrt{(b^2+x^2)}}{bx}+\frac{1}{c}\tan^{-1}\left(\frac{bx}{c\sqrt{(b^2+c^2+x^2)}}\right)\right]_0^a$$

$$=-\tfrac{1}{3}\left[\frac{\sqrt{(a^2+b^2)}}{ab}+\frac{1}{c}\tan^{-1}\left(\frac{ab}{c\sqrt{(a^2+b^2+c^2)}}\right)\right]+\infty=\infty.$$

DIOPHANTINE ANALYSIS.

113. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the four least integral numbers such that the difference of every two of them shall be a square number.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa., and J. A. SANDERS, Hackney, Ohio.

Let a, m^2+a, n^2+a, p^2+a be the numbers.

Then $p^2+a-a=p^2$, $n^2+a-a=n^2$, $m^2+a-a=m^2$, $p^2+a-n^2-a=p^2-n^2$, $p^2+a-m^2-a=p^2-m^2$, $n^2+a-m^2-a=n^2-m^2$.

Three of the conditions are satisfied and we must make p^2-n^2, p^2-m^2, n^2-m^2 all squares. This is done in Vol. IX, No. 4, pages 113-114.

$\therefore p=697d$, $n=185d$, $m=153d$.

$\therefore$ The numbers are a, $a+23409d^2$, $a+34225d^2$, $a+485809d^2$.

Let $a=d=1$, and the numbers are 1, 23410, 34226, 485810.

114. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the least integral values (if any) of a, b, and c that will make $2(a+b+c)\pm2\sqrt{[12ab-3(a+b-c)^2]}$ a square number for either sign of the radical.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This is always the case if $2(a+b+c)$ is the square, and $12ab=3(a+b-c)^2$ or $4ab=(a+b-c)^2$.

Let $a=(n+1)c$, $b=(n-1)c$.

Then $4ab=4(n^2-1)c^2$, $(a+b-c)^2=(2n-1)^2c^2$.

$\therefore 4(n^2-1)=(2n-1)^2=4n^2-4n+1$. $\therefore n=\frac{5}{4}$, $a=\frac{9}{4}c$, $b=\frac{1}{4}c$,

$\therefore 2(a+b+c)=7c$.

$7c=$a square if $c=7$, 7^3, 7^5, 7^7, etc.

$\therefore a=63$, $b=7$, $c=28$ are the least numbers, and $2(a+b+c)\pm2\sqrt{[12ab-3(a+b-c)^2]}=(14)^2$.

115. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Required the least three square integral numbers the difference between the sum of every two of them and the third shall be a square number.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x^2y^2, y^2z^2, z^2 be the numbers.

Then $x^2+y^2-1=a^2$.....(1), $x^2-y^2+1=b^2$.....(2). $\therefore x^2=\frac{1}{2}(a^2+b^2)$.

Let $a=t+u$, $b=t-u$. $\therefore x^2=t^2+u^2$.

Let $t=n(p^2-q^2)$, $u=2npq$. $\therefore x^2=n^2(p^2+q^2)^2$.

From (1), $y^2=a^2-x^2+1$.

$\therefore y^2=4n^2yq(p^2-q^2)+1=(2mn-1)^2$, suppose.

$$\therefore n=\frac{m}{m^2-pq(p^2-q^2)}.$$

Let $z=m^2-pq(p^2-q^2)$. Then $xz=m(p^2+q^2)$, $yz=m^2+pq(p^2-q^2)$.

$\therefore (x^2+y^2-1)z^2=m^2(p^2+2pq-q^2)^2$,

$(x^2-y^2+1)z^2=m^2(p^2-2pq-q^2)^2$,

$(y^2-x^2+1)z^2=2m^4+2p^2q^2(p^2-q^2)^2-m^2(p^2-q^2)^2$.

Let $m=p^2-q^2$. $\therefore (y^2-x^2+1)z^2=(p^2-q^2)^2(p^4+q^4-4p^2q^2)$.

Let $p^4+q^4-4p^2q^2=(p^2-2rq^2)^2$. $\therefore p^2=\dfrac{4r^2-1}{r-1}\cdot\dfrac{q^2}{4}$.

When $r=13$, $p^2=\frac{225}{16}q^2$, $m=\frac{209}{16}q^2$.

$\therefore xz=\frac{209}{256}(241)q^4$, $yz=\frac{209}{256}(269)q^4$, $z=\frac{209}{256}(149)q^4$.

Deleting the common factor $\frac{209}{256}$, $xz=241q^4$, $yz=269q^4$, $z=149q^4$, $(x^2+y^2-1)z^2=(329)^2q^8$, $(x^2-y^2+1)z^2=(89)^2q^8$, $(y^2-x^2+1)z^2=(191)^2q^8$.

Let $q=1$, and the least numbers all different are 241, 260, 149.

By giving q different values an infinite number of solutions can be found.

AVERAGE AND PROBABILITY.

137. Proposed by G. H. HARVILL, Malakoff, Texas.

A, B, C, and D, playing whist, agree that the person who first cuts an ace shall have a stake of \$313. What is the value of each person's expectation before the play begins, each taking his turn at cutting in the order named as the game progresses?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia. Pa., J. E. SANDERS, Hackney, Ohio, and J. SCHEFFER, A. M., Hagerstown, Md.

The chance that A cuts an ace$=\frac{1}{13}$, that he does not$=\frac{12}{13}$.
The chance that B cuts an ace$=\frac{1}{13}(\frac{12}{13})$, that he does not$=(\frac{12}{13})^2$.
The chance that C cuts an ace$=\frac{1}{13}(\frac{12}{13})^2$, that he does not$=(\frac{12}{13})^3$.
The chance that D cuts an ace$=\frac{1}{13}(\frac{12}{13})^3$, that he does not$=(\frac{12}{13})^4$; etc.

$\therefore A$'s chance$=\frac{1}{13}[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots]=\frac{2197}{7825}$.
B's chance$=\frac{1}{13}(\frac{12}{13})[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots]=\frac{2028}{7825}$.
C's chance$=\frac{1}{13}(\frac{12}{13})^2[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots]=\frac{1872}{7825}$.
D's chance$=\frac{1}{13}(\frac{12}{13})^3[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots]=\frac{1728}{7825}$.

A's expectation$=\frac{2197}{7825}$ of \$313=\$87$\frac{6886}{7825}$;
B's expectation$=\frac{2028}{7825}$ of \$313=\$81$\frac{939}{7825}$;
C's expectation$=\frac{1872}{7825}$ of \$313=\$74$\frac{6886}{7825}$;
D's expectation$=\frac{1728}{7825}$ of \$313=\$69$\frac{939}{7825}$.

138. Proposed by G. B. M. ZERR, A. M.. Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

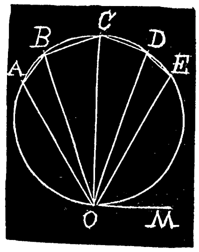
Find the average area of, (1) triangle, (2) quadrilateral, (3) pentagon, (4) hexagon, formed by taking, (1) three, (2) four, (3) five, (4) six random points on the circumference of a given circle radius a.

Solution by J. E. SANDERS, Hackney. Ohio, and the PROPOSER.

We will solve for hexagon first. Let $\angle AOM=\theta$, $\angle BOM=\phi$, $\angle COM=\psi$, $\angle DOM=\rho$, $\angle EOM=\delta$.

Then $AO=2a\sin\theta$, $BO=2a\sin\phi$, $CO=2a\sin\psi$, $DO=2a\sin\rho$, $EO=2a\sin\delta$, where a=radius of circle.

Area of hexagon$=u=2a^2[\sin\theta\sin\phi\sin(\theta-\phi)+\sin\phi\sin\psi\sin(\phi-\psi)+\sin\psi\sin\rho\sin(\psi-\rho)+\sin\rho\sin\delta\sin(\rho-\delta)]$.



$$\therefore \triangle = \frac{\int_0^\pi \int_0^\theta \int_0^\phi \int_0^\psi \int_0^\rho u d\theta\, d\phi\, d\psi\, d\rho\, d\delta}{\int_0^\pi \int_0^\theta \int_0^\phi \int_0^\psi \int_0^\rho d\theta\, d\phi\, d\psi\, d\rho\, d\delta}$$

$$= \frac{120}{\pi^5} \int_0^\pi \int_0^\theta \int_0^\phi \int_0^\psi \int_0^\rho u d\theta\, d\phi\, d\psi\, d\rho\, d\delta$$

$$= \frac{120a^2}{\pi^5} \int_0^\pi \int_0^\theta \int_0^\phi \int_0^\psi [2\rho \sin\theta \sin\phi \sin(\theta-\phi)$$

$$+2\rho \sin\phi \sin\psi \sin(\phi-\psi) + 2\rho \sin\psi \sin\rho \sin(\psi-\rho)$$

$$+\sin^2\rho - \rho\sin\rho\cos\rho] d\theta\, d\phi\, d\psi\, d\rho \ldots\ldots (1),$$

$$= \frac{30a^2}{\pi^5} \int_0^\pi \int_0^\theta \int_0^\phi [4\psi^2 \sin\theta \sin\psi \sin(\theta-\phi) + 4\psi^2 \sin\phi \sin\psi \sin(\phi-\psi) + 3\psi$$

$$-3\sin\psi \cos\psi - 2\psi^2 \sin\psi \cos\psi] d\theta\, d\phi\, d\psi \ldots\ldots (2),$$

$$= \frac{20a^2}{\pi^5} \int_0^\pi \int_0^\theta [2\phi^3 \sin\theta \sin\phi \sin(\theta-\phi) + 3\phi^2 - \phi^3 \sin\phi \cos\phi - 3\sin^2\phi] d\theta\, d\phi \ldots\ldots (3),$$

$$= \frac{5a^2}{2\pi^5} \int_0^\pi [10\theta^3 - 15\theta - 2\theta^4 \sin\theta \cos\theta + 15 \sin\theta \cos\theta] d\theta = \frac{15a^2}{2\pi^5}\left(1 - \frac{3}{\pi^2}\right).$$

For the pentagon the points can be taken $\frac{1}{5}$ the number of ways for a hexagon. Writing θ for ϕ in the three last terms of (3), we get for the pentagon,

$$\triangle = \frac{4a^2}{\pi^4} \int_0^\pi [3\theta^2 - \theta^3 \sin\theta \cos\theta - 3\sin^2\theta] d\theta = \frac{5a^2}{\pi^4}\left(1 - \frac{3}{2\pi^2}\right)$$

For the quadrilateral, the points can be taken $\frac{1}{20}$ the number of ways for a hexagon. Writing θ for ψ in the last three terms of (2), we get,

$$\triangle = \frac{3a^2}{2\pi^3} \int_0^\pi [3\theta - 3\sin\theta \cos\theta - 2\theta^2 \sin\theta \cos\theta] d\theta = \frac{3a^2}{\pi}.$$

For the triangle the points can be taken $\frac{1}{60}$ the number of ways for a hexagon. Writing θ for ρ in the last two terms of (1), we get

$$\triangle = \frac{2a^2}{\pi^2} \int_0^\pi [\sin^2\theta - \theta \sin\theta \cos\theta] d\theta = \frac{3a^2}{2\pi}.$$

These last three follow at once from the fact that for the pentagon, quadrilateral, and triangle, we have three, two, and one term in the expression for the area, while the power of π is for the pentagon π^4, for the quadrilateral π^3, for the triangle π^2, in the denominator.

139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

Remark by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

No. 139 is the same as No. 130 for which I have sent a solution previously.

140. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Obtain the average area of a triangle formed by a tangent to the four—cusped hypocycloid and the coördinate axes.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a=radius of fixed circle. Then portion of tangent intercepted by coordinate axes=a. Area of triangle=$\frac{1}{2}xy$, subject to the condition $x^2+y^2=a^2$.

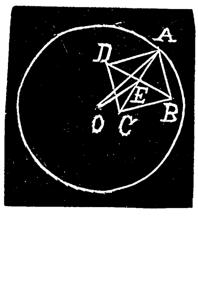
$$\therefore \text{Average area} = \tfrac{1}{2}\int_0^a x\sqrt{(a^2-x^2)}\,dx \Big/ \int_0^a dx = \tfrac{1}{6}a^2.$$

141. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Upon a circular table, radius r, a *variable* square plate is thrown at random. What is the probability that the plate will lie wholly on the table?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the given circle, radius r; $ABCD$ the square, center E; $AB=2x$, $OE=z$, $\angle OAE=\theta$. Then $AE=x\sqrt{2}$.



If the center of the square, E, falls on a circle center O and radius $(r-x\sqrt{2})$, the square will be wholly on the table. If E falls on a circle, center O and radius z, the plate will lie wholly on the table. $z=\sqrt{[r^2+2x^2-2rx\sqrt{(2)}\cos\theta]}$. The limits of x are 0 and $\frac{1}{2}r\sqrt{a}$; of θ, 0 and $\frac{1}{4}\pi-\sin^{-1}(x/r)=\theta'$. Let p=chance. Since the whole number of ways E can fall on the circle is πr^2, we get,

$$p=\frac{\pi\int_0^{\frac{1}{2}r\sqrt{2}}[r-x\sqrt{2}]^2dx}{\pi r^2\int_0^{\frac{1}{2}r\sqrt{2}}dx}+\frac{\pi\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta'}z^2dx\,d\theta}{\pi r^2\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta'}dx\,d\theta}$$

$$=\frac{\sqrt{2}}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}[r-x\sqrt{2}]^2dx+\frac{[2+\sqrt{2}]}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta'}z^2dx\,d\theta$$

$$=\tfrac{1}{3}+\frac{[2+\sqrt{2}]}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}\{[r^2-2x^2][\tfrac{1}{4}\pi-\sin^{-1}\frac{x}{r}]-2x[\sqrt{(r^2-x^2)}-x]\}dx=1-\frac{\sqrt{2}}{9}.$$

MISCELLANEOUS.

136. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

(1) Solve (to five places) the equations, $\sin(x+\frac{1}{6}\pi)=10\sin x$, and $a\cos\phi\log\sin\phi=p$ where a is small and positive, and $\varphi=\alpha+\kappa$, where κ is very small and α is not very small; (2) If $a\theta=b\varphi$ where a is prime to b, and $\sin\theta=p$, $\sin\varphi=q$, how many values of q are there for each of p ? (3) if $2x=\sin^{-1}x$, show there is only one positive value of x, and find it.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) $\sin(x+\frac{1}{6}\pi)=\frac{1}{2}\sqrt{3}\sin x+\frac{1}{2}\cos x=10\sin x$.

$\therefore(20-\sqrt{3})\sin x=\cos x$, or $\tan x=\dfrac{1}{20-\sqrt{3}}=\dfrac{20+\sqrt{3}}{397}=.05474$. $\therefore x=3°8'$.

$\cos\varphi\log\sin\varphi=p/a$ or $\log(\sin\varphi)^{\cos\phi}=p/a$.

$\therefore\sin^2\varphi=e^{2p\sec\phi/a}$.

$$1-\cos^2\varphi=1+\frac{2p}{a}\sec\varphi+\frac{4p^2}{2!a^2}\sec^2\varphi+\frac{8p^3}{3!a^3}\sec^3\varphi+.....$$

$$-a/2p=\sec^3\varphi+\frac{p}{a}\sec^4\varphi+\frac{2p^2}{3a^2}\sec^5\varphi+\frac{p^3}{3a^3}\sec^6\varphi+.....$$

By reversion of series,

$$\sec\varphi=-\left(\frac{a}{2p}\right)^{\frac{1}{3}}-\frac{1}{3}\left(\frac{p}{4a}\right)^{\frac{1}{3}}-\frac{p}{18a}-\frac{p^2}{81a^2}\left(\frac{a}{2p}\right)^{\frac{1}{3}}-\frac{29p^2}{2430a^2}\left(\frac{p}{4a}\right)^{\frac{1}{3}}.....$$

The value of $\sec\varphi$ to five terms. Since a is small, p is still smaller. When any multiple of $\frac{1}{2}\pi$ is substituted for φ, $\cos\varphi\log\sin\varphi$ is a maximum.

(2) $\sin\theta=\sin(m\pi-\pi)=\sin(2m\pi+\theta)=p$,

$\sin\varphi=\sin(m\pi-\varphi)=\sin(2m\pi+\varphi)=q$.

Since $a\theta=b\varphi$, but one value of q satisfies one value of p; for $a(n\pi\pm\theta)=b(n\pi\pm\varphi)$ only when $a=b$ and $a\theta=b(n\pi\pm\varphi)$ never.

(3) $x=\sin 2x=2x-\dfrac{8x^3}{3!}+\dfrac{32x^5}{5!}-\dfrac{128x^7}{7!}+\dfrac{512x^9}{9!}+\ldots$

$\therefore y=0$ or $1=\dfrac{8x^2}{3!}-\dfrac{32x^4}{5!}+\dfrac{128x^6}{7!}-\dfrac{512x^8}{9!}+\dfrac{2048x^{10}}{11!}-\ldots$

By reversion of series, $x=.94775$, nearly.
$2x=108°\ 36'$, nearly. $x=54°\ 18'$, nearly.

137. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

The first transvectant of the binary cubic and its second transvectant is the *cubicovariant* of the binary cubic.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and W. G. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let $u=ax^3+3bx^2y+3cxy^2+dy^3$, be the binary cubic. Then the second transvectant is

$$\frac{d^2u}{dx^2}\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dxdy}\right)^2=0=(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2.$$

This is the Hessian of the cubic.
Let $v=(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2$.
The first transvectant of u, v is

$$\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}=0.$$

$$\therefore 3(ax^2+2bxy+cy^2)[ad-bc)x+2(bd-c^2)y]$$
$$-3(bx^2+2cxy+dy^2)[(ad-bc)y+2(ac-b^2)x]=0.$$

$$\therefore (a^2d-3abc+2b^3)x^3+3(abd-2ac^2+b^2c)x^2y$$
$$+3(2b^2d-bc^2-acd)xy^2+(3bcd-ad^2-2c^3)y^3=0.$$

This is the cubicovariant of the cubic u.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

170. Proposed by J. F. LAWRENCE, A. B., Breckenridge, Mo.

Suppose the market value of 5% bank stock to be $11\frac{1}{9}$% higher than 8% corporation bonds; I realize 8% on my investment, and my income from each is \$180; what did I invest in each?

171. Proposed by JOHN S. ROYER, Editor of The School Visitor, Columbus, Ohio.

A drawer made of inch boards is 8 inches wide, 6 inches deep, and slides horizontally. How far must it be drawn out to put into it a book 4 inches thick, 6 inches wide, and 9 inches long?

ALGEBRA.

183. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the condition that $x:y:z$ may be real, given that $\Sigma ax^2=\Sigma a\alpha x=0$, and $\Sigma a\alpha^2=1$.

184. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

If m rows, viz., the h_1th, h_2th, ..., h_mth, be transferred so as to become the 1st, 2d, ..., mth, without altering the relative positions of the remaining rows, and that n columns, viz., the k_1th, k_2th, ..., k_nth, be similarly transformed, the determinant thus obtained is the same as the original or differs from it only in sign according as $h_1+h_2+.....+h_m-\frac{1}{2}m(m+1)+k_1+k_2+.....k_n-\frac{1}{2}n(n+1)$ is odd or even. [*Muir.*]

GEOMETRY.

205. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Having given any two systems of conjugate semi-diameters of an ellipsoid, the parallelopiped which has any three for continuous edges is equal to that which has the other three for continuous edges.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$ABCD$ is circumscribed by a circle center O, and it circumscribes a circle radius r. The perpendiculars from C on the sides are x, y, z, u. Show that $\frac{1}{2}AC.BD=r\Sigma x$.

CALCULUS.

170. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find the center-locus of conics having 4-point contact with a given conic at a given point. Show that the conic of minimum eccentricity is given by $e^4ab^2\varphi+4e^2-4=0$, where e is its eccentricity, and φ is the angle which the linear center-locus above makes with the normal to the curve at the point.

171. Proposed by J. E. SANDERS, Hackney, Ohio.

A thread passes spirally around a *rough* cylinder 10 feet high and 6 inches in diameter. How far will a pigeon fly in unwinding the thread if the distance between the coils is 4 inches, and the thread *unwound* is at all times *horizontal?*

MECHANICS.

161. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Four equal uniform smoothly jointed rods length a, and weight w, form a rhombus $ABCD$, A and C being in contact with two vertical walls b feet apart. An elastic string, natural length x, modulus λ, keeps the figure in position. The angle of friction at A and C is $\tan^{-1}p$. When the rhombus is just about to slip, find the angle A, and the angle between AB and the vertical.

162. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Show that the velocity, v, of a wave along the surface of a liquid whose depth is not less than the wave-length, λ, whose density is δ, and surface tension, T, is $v^2=\dfrac{g\lambda}{2\pi}+\dfrac{2\pi T}{\lambda\delta}$.

DIOPHANTINE ANALYSIS.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least integral numbers such that their sum shall be a square, and the sum of their squares a biquadrate.

AVERAGE AND PROBABILITY.

147. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud; England.

In a bag are n balls, known to be black or white, either color, *a priori*, equally likely. I draw two, which turn out to be one white and one black. I replace them and draw two more. What is the chance both are black?

NOTES.

Professor Lon C. Walker has been elected Professor of Mathematics in the Colorado School of Mines, Golden, Colorado.

Dr. Saul Epsteen, of the University of Chicago, has very kindly consented to edit, for the coming year, the problems and solutions for the MONTHLY. So beginning with this issue all problems and solutions should be sent to him. The MONTHLY is to be congratulated that it has fallen into the hands of such able mathematicians as Drs. Dickson and Epsteen. We predict for it the most prosperous year in its history. B. F. F.

ERRATA.

On page 171, problem 105, for "$a^{2m}+b^{2m}$" read $a^{2^m}+b^{2^m}$.

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ON THE GROUPS OF THE FIGURES OF ELEMENTARY GEOMETRY.

By PROFESSOR G. A. MILLER.

Among the various elementary illustrations of the group concept those which relate to the movements of space are perhaps the most instructive. Even those students who have no trouble in approaching the subject by analytic methods generally take pleasure in observing the geometric interpretation of some of the most useful groups. In what follows we shall aim to determine the groups of the most common geometric figures rather than to make use of these figures for the sake of illustrating known groups. Only the most elementary notions about group theory are presupposed. Only groups of finite order are here considered.

The group of a figure is composed of all the movements of space which transform the figure into itself. In these movements space is regarded as rigid, that is, any two points are transformed into two points at the same distance from each other as the original points.

All the movements which transform a system of points into itself must also transform a certain point into itself. This elementary fact may be proved as follows: There is at least one minimum sphere which includes all these points. If there were two distinct minima spheres having this property, all the given points would be in the space common to the two spheres. This is impossible since this common space is in a smaller sphere; viz., in the one whose radius is equal to the radius of the circle which is composed of the points common to the two equal spheres. Every movement which transforms a system of points into itself must therefore transform the center of the minimum circumscribing

sphere into itself. Hence such movements may be composed of rotations around this center and ever concentric sphere must be transformed into itself.

We shall first consider the rotations which transform a plane triangle into itself. If the triangle is equilateral there are three such rotations around the axis perpendicular to its plane at the center; viz., the rotations through 0°, 120°, and 240°, respectively. These three rotations clearly form the group of order 3. This, however, is not the group of the triangle, for there are three other rotations which transform it into itself; viz., the rotations around the lines of symmetry through 180°. The group formed by these six rotations (the equilateral triangle group) is evidently simply isomorphic with the group formed by all the possible permutations of three things,—the symmetric group of degree 3. It may be observed that any two of these rotations of period two are equivalent to one of period three and that the rotations of period two are non commutative; i. e., the rotation of period three which is equivalent to two rotations of period two depends upon the order in which they are taken. Hence the equilateral triangle group is non-commutative. It will be found that all the groups of the regular plane polygons and of the regular solids are non-commutative. All the groups of the spherical polygons are commutative. In fact, they are always cyclic.

If a triangle is isosceles without being also regular, its group is clearly of order two; while the identity is the group of the scalene triangle. The cyclic group of order 3 is the group of the equilateral spherical triangle. All other spherical triangles evidently admit no rotation besides the identity. Hence we see that *the group of the plane equilateral triangle is of order 6, that of the spherical equilateral triangle is of order 3, that of the plane isosceles triangle is of order 2. The other triangles admit no rotations besides the identity.*

Before considering the groups of movements of the quadrangles it may be desirable to consider the subject from a somewhat more general standpoint. Either all or just half of the movements which transform into itself a system of co-planar points finite in number consist of rotations around an axis perpendicular to the plane through the center of the minimum enclosing circle. Among these rotations there must be a smallest one and all the others are composed of repetitions of this smallest rotation. Hence all such rotations form a cyclic group. If there is any additional rotation it must be of period two. If this rotation is followed by each of the rotations of the given cyclic subgroup, in order, we obtain just as many distinct rotations of order two as there are operations in this subgroup. A group which contains a cyclic subgroup of half its order and only operations of order two besides those in this subgroup is known as a dihedral rotation group. *All the groups of movements of systems of co-planar points, finite in number, must therefore be either cyclic or of the dihedral rotation type.* It is easy to see that the dihedral rotation group of order $2n$ is the group of the regular plane polygon of n sides. On this account the dihedral rotation groups may be called the regular polygon groups. If s is any operation in the cyclic subgroup of order n and if t is any other operation of the group it follows that $st=$ some t and hence $stst=1$. Hence $t^{-1}st=s^{-1}$; that is, t transforms every s into its

inverse. This result can also be seen geometrically and a dihedral rotation group may be defined by the fact that it is composed of a cyclic group of order n and n operations of order two transforming each operation of this cyclic subgroup into its inverse. The regular polygon group of order $2n$ is therefore non-commutative whenever n exceeds two.

It is now easy to determine all the quadrangle groups. The largest of these is the group of the square. This consists of the cyclic group of order 4 and four additional operations of order 2 corresponding to the rotations around its four lines of symmetry. Since each of these eight rotations, with the exception of the identity, permutes at least two of the vertices, the square group can be represented on four letters and hence it is simply isomorphic with the octic group.* It contains two non-cyclic subgroups of order 4. The transitive one is the group of a rectangle and the intransitive one is the group of a rhombus. The other subgroup of order 4 is cyclic and it is the group of the spherical quadrangle whose vertices determine a square. All the other quadrangles belong either to the group of period 2 or to the identity.

It is evident that a crossed quadrangle whose vertices coincide with those of a square or rectangle, belongs to the group of order four. Since crossed polygons are not generally studied in elementary geometry we shall exclude them from our consideration. There are two classes of plane convex quadrangles whose group is of order 2; viz., (1) Those whose diagonals intersect at right angles, only one of them being bisected, (2) Those whose unequal diagonals bisect each other obliquely, the general parallelograms. The latter are the only plane quadrangles which permit a non-identical movement without having any line of symmetry.

If a quadrangle is concave, its group can differ from the identity only when the concave vertex is at the center of the circle circumscribing the other three vertices and when these vertices determine an isosceles triangle. As the group of such a quadrangle is of order two, there are just three classes of plane quadrangles which have this group. A spherical quadrangle belongs to this group if the common perpendicular of the lines joining two pairs of vertices bisects each of these lines, and the vertices of the quadrangle do not determine a square. All the other quadrangles admit no movement besides the identity.

There are only two substitution groups on three letters. These are the groups of movements of the plane and the spherical equilateral triangles. There are seven groups on four letters. Of the latter, the group of order 2 is the group of movements of various quadrangles, the three groups of order 4 are the groups of movements of the rectangle, rhombus, and spherical quadrangle whose vertices determine a square, respectively; and the octic group is the group of movements of the square. As the other two possible substitution groups of degree 4 are neither cyclic nor of the dihedral rotation type they cannot be the groups of movements of any polygons. It is, however, easy to verify that they are the groups of movements of the regular tetrahedron and cube respectively. Hence

*Pierpont, Annals of Mathematics, Vol. 1, 1900, page 140.

each one of the seven possible substitution groups on four letters is a group of movements.

Comparatively few of the groups whose degrees exceed 4 are groups of movements. The most interesting of these is the icosaedron group, which is simply isomorphic with the alternating group on five letters, and is the group of movements of the regular icosaedron and also of the regular dodecaedron. The regular octaedron has the same group of movements as the cube; viz., the symmetric group on four letters, while the regular tetraedron admits only the rotations corresponding to the alternating group on four letters. The orders of these groups can be easily found by observing the number of possible rotations of the solid which transform a face into itself and multiplying this number by the number of faces.

If the base of a regular pyramid has n sides ($n \neq 3$) its group is the cyclic group of order n. This is also the case when $n=3$ and the pyramid is not a regular tetraedron. If the base of a regular prism has n sides ($n \neq 4$) its group is the dihedral rotation group of order $2n$. When $n=4$ and a lateral edge is not equal to a base edge the group of the regular prism is also the dihedral rotation group of order $2n$. The rectangular parallelopiped with three unequal edges belongs to the rectangle group and it is easy to construct pyramids which belong to the group of order two. Hence *all the possible groups of movements of polygons are also groups of movements of either the pyramids or the prisms.*

It has been observed above that solids may have three additional groups of movements. That is, there are three groups of rotation which are neither cyclic nor of the dihedral rotation type. The proof that there are no more than three such groups is not very difficult. It is only necessary to observe that each complete set of conjugate vertices must have the same group and must lie on a certain sphere, whenever the number of these vertices exceeds two. Hence we may confine our attention to any one complete set of conjugate vertices provided it includes more than two.*

If a complete set of conjugate vertices (which may consist of just one vertex) lies on a plane which does not pass through the center of the minimum circumscribed sphere, the group of the solid is cyclic. When this condition is not satisfied and a complete set of conjugate vertices lies on two parallel planes the group of the solid must be metacyclic.

In all other cases the points where the conjugate axes of rotations meet the surface of the sphere on which the complete set of conjugate vertices lies must form the vertices of a regular polyhedron.† As the number of the former points must exceed two, their group is identical with the group of the vertices. These results may be stated as follows: *If the group of movements of a system of points is neither cyclic nor of the dihedral rotation type, it must be the tetraedron, octahedron, or the icosaedron group.*

*If each conjugate set is composed of only two vertices it is easy to see that the group must be either of order two or of order four.

†For a more complete proof see Jordan, *Annali di Matematica*, Vol. 2, 1868; also Klein, *Ueber das Ikosaeder*, 1884, page 21.

THREE ALGEBRAIC NOTES.

By DR. L. E. DICKSON.

I. Transformation of a Solvable Quintic into Itself.

As well known, the following quintic equation

$$(1) \qquad y^5+py^3+\tfrac{1}{5}p^2y+r=0$$

may be solved algebraically by making the substitution

$$y=z-\tfrac{1}{5}p/z,$$

quite analogous to Cartan's solution of a reduced cubic equation.

We here consider another property of (1), which may be written

$$(1') \qquad y\{(y^2+\tfrac{1}{2}p)^2-\tfrac{1}{20}p^2\}+r=0.$$

This form suggests the following quadratic transformation

$$y^2+\tfrac{1}{2}p=w,\ \ y=\surd(w-\tfrac{1}{2}p).$$

After rationalization, (1′) then becomes

$$(w-\tfrac{1}{2}p)(w^2-\tfrac{1}{20}p^2)^2=r^2,$$

a solvable quintic containing al powers $\overline{<}5$ of w. To remove the term w^4, set $w=x+\frac{1}{10}p$. We thus obtain

$$(x-\tfrac{2}{5}p)(x^2+\tfrac{1}{5}px-\tfrac{1}{25}p^2)^2=r^2,$$

$$(2) \qquad x^5-\tfrac{1}{5}p^2x^3+\tfrac{1}{125}p^4x-\frac{2}{5^5}p^5-r^2=0,$$

a quintic lacking the term x^2 as well as x^4. Setting

$$Y=ax,\ \ P=-\tfrac{1}{5}p^2a^2,\ \ R=-\frac{2}{5^5}p^5a^5-r^2a^5,$$

equation (2) takes the following form, analogous to (1),

$$(3) \qquad Y^5+PY^3+\tfrac{1}{5}P^2Y+R=0.$$

Now equations (1) and (3) are identical in exactly two cases:

$$a=\sqrt{\frac{-5}{p}},\ \ r=\sqrt{\left(\frac{-p}{5}\right)^5}\ ;\ \ a=-\sqrt{\frac{-5}{p}},\ \ r=-2\sqrt{\left(\frac{-p}{5}\right)^5}.$$

Theorem. *According as* $r=[\sqrt{(-p/5)}]^5$ *or* $-2[\sqrt{(-p/5)}]^5$, *the substitution* $y^2=Y\sqrt{(-p/5)}-\frac{2}{5}p$, *or* $y^2=-Y\sqrt{(-p/5)}-\frac{2}{5}p$ *transforms the quintic equation* (1) *into itself.*

In the second case, $\frac{r^2}{4}+\left(\frac{p}{5}\right)^5=0$, so that the quintic has two pairs of equal roots. In fact (Dickson's *College Algebra*, page 189), the roots are

$$2\sqrt{(-p/5)},\ \text{two each } \alpha\sqrt{(-p/5)},\ \text{two each } \beta\sqrt{-p/5}),$$

where $\alpha=\epsilon+\epsilon^4$, $\beta=\epsilon^2+\epsilon^3$ are the roots of $w^2+w-1=0$; also $\epsilon^5=1$.

II. Reducibility of a Reciprocal Sextic Equation.

The general reciprocal equation of the sixth degree may be written

$$x^6-px^5+qx^4-rx^3+qx^2-px+1=0. \tag{4}$$

As known, its solution may be made to depend upon a cubic and a quadratic equation; thus, set

$$x+\frac{1}{x}=z,\ \text{whence}\ x^2+\frac{1}{x^2}=z^2-2,\ x^3+\frac{1}{x^3}=z^3-3z. \tag{5}$$

Dividing (4) by x^3, we then obtain

$$z^3-pz^2+(q-3)z+2p-r=0. \tag{6}$$

The question here proposed is the determination of the conditions under which (4) has a factor of specified form and with coefficients in an initially given domain of rationality. Some examples will make clear what is meant by a *domain*. It is a familiar theorem that, when p is a prime number, $x^{p-1}+\dots+x^2+x+1$ is algebraically irreducible, not being expressible as a product of two factors each of the form $f=ax^n+bx^{n-1}+\dots+t$, where $1\leqq n<p-1$, and a, b, ..., t are *integers*. It readily* follows that there is no factor f with *rational* coefficients. Hence $x^{p-1}+\dots+x^2+x+1$ is irreducible in the *domain of all rational numbers*. However, it may become reducible in a larger domain. Thus, for $p=5$,

$$x^4+x^3+x^2+x+1=[x^2+\tfrac{1}{2}x(1+\sqrt{5})+1][x^2+\tfrac{1}{2}x(1-\sqrt{5})+1],$$

and hence is reducible in the domain of all rational functions of $\sqrt{5}$ with integral coefficients. It becomes completely reducible into linear factors in a domain containing an imaginary fifth root of unity.

We return to the question of the reducibility of (4) in a given domain or field F containing the coefficients of p, q, r. The condition that (4) shall have a

*For one proof, with references to the various proofs, of these two statements, see the writer's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons, 1903), page 76.

linear factor $x-x_1$ is that the cubic (6) shall have a root z_1 in the domain F such that $x^2-z_1x+1=0$ has a root x_1 in F, the latter requiring that z_1^2-4 be the square of an element of F. With one linear factor $x-x_1$ is associated a second, viz., $x-1/x_1$. The condition for at least three linear factors is that (6) shall have at least two and hence all its roots z in F, such that z^2-4 be a square in F.

From the above method of solving (4), or by undetermined coefficients, it follows that (4) has a factor of the form x^2-z_1x+1 if and only if equation (6) has a root z_1 in F. Next, if (4) has a factor x^2-kx+t, where $t\neq 0$, $t\neq 1$, it has also the factor $t^{-1}(tx^2-kx+1)=x^2-t^{-1}kx+t^{-1}$, and hence a third factor x^2-zx+1. The condition for this third factor was that (6) have a root z in F. Let this condition be satisfied. Then the quotient of (4) by x^2-zx+1 is the reciprocal quartic equation with coefficients in F:

(7) $$x^4+(z-p)x^3+(z^2-zp-1+q)x^2+(z-p)x+1=0.$$

We are therefore led to consider the conditions for the factorization

(8) $$x^4-ax^3+bx^2-ax+1=(x^2-kx+t)(x^2-kt^{-1}x+t^{-1}).$$

The conditions are

(9) $$a=k(1+t^{-1}),\quad b=t+t^{-1}+k^2t^{-1}.$$

Eliminating* k, we obtain the reciprocal condition on t

(10) $$t^4+(2-bt^3+(a^2-2b+2)t^2+(2-b)t+1=0.$$

If (10) has a root $t\neq -1$ in F, the first condition (9) determines k in F. If every root of (10) equals -1, then $a=0$, $b=-2$, $k=0$, so that the factorization (8) is evident. For the general case, we may readily determine the conditions under which (10) has a root in F. First, it is necessary that $Z\equiv t+t^{-1}$ belong to F. By (10),

$$Z^2-(2-b)Z+a^2-2b=0,\quad (Z-1+\tfrac{1}{2}b)^2=(1+\tfrac{1}{2}b)^2-a^2.$$

Hence $(1+\frac{1}{2}b)^2-a^2$ must be the square of an element l of F. Next, from $t^2-tZ+1=0$, $Z^2-4\equiv\frac{1}{2}b^2-2-a^2+l(2-b)$ must be a square in F.

To proceed otherwise, let z_1 and z_2 be the roots of $z^2-az+b-2=0$. Then

$$x^4-ax^3+bx^2-ax+1=(x^2-z_1x+1)(x^2-z_2x+1).$$

Setting $Z_1=\sqrt{(z_1^2-4)}$, $Z_2=\sqrt{(z_2^2-4)}$, the four roots are

$$\tfrac{1}{2}z_1+\tfrac{1}{2}Z_1,\quad \tfrac{1}{2}z_1-\tfrac{1}{2}Z_1,\quad \tfrac{1}{2}z_2+\tfrac{1}{2}Z_2,\quad \tfrac{1}{2}z_2-\tfrac{1}{2}Z_2.$$

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*Upon eliminating t, there results a non-reciprocal quartic for k.

The two roots of either of the quadratic factors (8) are not reciprocal. By proper choice of the signs of Z_1 and Z_2, we may assume that $\frac{1}{2}z_1+\frac{1}{2}Z_1$ and $\frac{1}{2}z_2+\frac{1}{2}Z_2$ are the roots of one of the quadratic factors (8) and the remaining two the roots of the other. Now the coefficients of a quadratic (8) belong to F if and only if the sum and product of its two roots do. But $z_1+z_2=a$ and $z_1z_2=b-2$ belong to F. The conditions are therefore that

$$Z_1+Z_2,\ z_1Z_2+z_2Z_1+Z_1Z_2,\ -z_1Z_2-z_2Z_1+Z_1Z_2$$

shall belong to F. It follows first that $Z_1Z_2=2\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$ shall belong to F, say $=-2l$. Next, $z_1Z_2+z_2Z_1$ must belong to F. Its square equals

$$2z_1^2z_2^2-4z_1^2-4z_2^2+2z_1z_2Z_1Z_2=4[\tfrac{1}{2}b^2-2-a^2+l(2-b)].$$

Hence $\frac{1}{2}b^2-2-a^2+l(2-b)$ must be a square in F. We have now obtained the two conditions given earlier. I remains to verify that Z_1+Z_2 belong to F. But

$$(z_1+z_2)(Z_1+Z_2)=(z_1Z_1+z_2Z_2)+(z_1Z_2+z_2Z_1),$$

$$(z_1Z_1+z_2Z_2)(z_1Z_2+z_2Z_1)=z_1z_2(Z_1^2+Z_2^2)+Z_1Z_2(z_1^2+z_2^2),\ \text{in } F.$$

Consider finally the most interesting case of all, that in which the sextic (4) has an irreducible cubic factor. Call its roots κ, λ, μ. No two of the three are reciprocal, since the product of all three would then equal the third root, which would thus belong to F, making the cubic reducible. Hence κ^{-1}, λ^{-1}, μ^{-1} are the roots of a second cubic factor of (4). Hence (4) is the product of the two cubic factors

$$(11)\qquad x^3+ax^2+bx+c,\ x^3+\frac{b}{c}x^2+\frac{a}{c}x+\frac{1}{c}.$$

Equating the coefficients of the product to those of (4), we get

$$(12)\qquad a+\frac{b}{c}=-p,\ b+\frac{a\,b}{c}+\frac{a}{c}=q,\ c+\frac{1}{c}+\frac{b^2}{c}+\frac{a^2}{c}=-r.$$

Substituting the value of a from the first in the last two, we get

$$(13)\qquad b^2+b(1+pc-c^2)+pc+qc^2=0,$$

$$(14)\qquad b^2(1+c^2)+2bpc+p^2c^2+c^2+c^4+rc^3=0.$$

Eliminating b^2, we get $bU+W=0$, where

$$(15)\qquad U=c^4-pc^3+pc-1,\ W=p^2c^2+c^2+c^4+rc^3-pc-pc^3-qc^2-qc^4.$$

Substituting $b=-W/U$ in (13) and cancelling the factor c^2, we get

$$(16)\quad c^8+1+(c^7+c)(r-2p)+(c^6+c^2)(q^2-2pr+2p^2+1-2q)$$
$$+(c^5+c^3)(r-2p)(p^2-2q+1)+c^4(2q^2+r^2+p^4+4p^2-4q-4p^2q)=0,$$

which is evidently a reciprocal equation. Making the substitution, similar to (5),

$$(17)\quad c+\frac{1}{c}=y,\ c^2+\frac{1}{c^2}=y^2-2,\ c^3+\frac{1}{c^3}=y^3-3y,\ c^4+\frac{1}{c^4}=y^4-4y^2+2,$$

equation (16) becomes

$$(18)\quad y^4+y^3(r-2p)+y^2(q^2-2pr+2p^2-3-2q)$$
$$+y(r-2p)(p^2-2q-2)+p^4-4p^2q+4pr+r^2=0.$$

Suppose that y and then c can be determined in F. Now

$$(19)\quad U=(c^2-1)(c^2-pc+1).$$

If $U\neq 0$, then b and a are uniquely determined by $bU+W=0$ and the first relation (12). Next, $U=0$ if and only if $c=\neq 1$ or $c^2+1=cp$, whence $y=2,\ -2$, or p. If $c=1$, $W=0$ gives $r-2q+p^2-2p+2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2+bp+p+q=0$. If $c=-1$, $W=0$ gives $r+2q-2p-p^2-2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2-bp-p+q=0$. If $c^2=pc-1$, whence $y=p$, then $W=0$ gives $r+p-pq=0$, and (18) reduces to $(r+p-pq)^2=0$, while (13) and (14) reduce $b^2+2b+qc^2+pc=0$.

In general, the values of a and b follow rationally when c is found from the quartic and quadratic. In the three cases mentioned, c is given, but b requires the solution of a quadratic.

III. The General Term of a Recurring Series.

The usual examples under this topic lead to a generating fraction which can be resolved into partial fractions with binomial denominators, so that the general term of the expansion into series is readily found. It is otherwise with the following example (No. 4, page 148, Dickson's *College Algebra*):

$$(20)\quad 1+6x-4x^2-40x^3-112x^4-32x^5+704x^6+\ldots$$

The generating relation is $u_n-2xu_{n-1}+4x^2u_{n-2}+8x^3u_{n-3}=0$; the generating fraction is

$$(21)\quad (1+4x-12x^2)/D,\ D\equiv 1-2x+4x^2+8x^3.$$

First method. Set $D=(1-2\alpha x)(1-2\beta x)(1-2\gamma x)$. Then

$$(22)\quad \alpha+\beta+\gamma=1,\ \alpha\beta+\alpha\gamma+\beta\gamma=1,\ \alpha\beta\gamma=-1.$$

The discriminant $\triangle \equiv (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ is found to be -44. Now

$$\frac{1}{D}=\frac{a}{1-2\alpha x}+\frac{b}{1-2\beta x}+\frac{c}{1-2\gamma x},$$

$$1=a\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)=b\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\gamma}{\beta}\right)=c\left(1-\frac{\alpha}{\gamma}\right)\left(1-\frac{\beta}{\gamma}\right).$$

$$\therefore \frac{-\triangle}{D}=\frac{\alpha^2(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha)}{1-2\alpha x}+\frac{\beta^2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2}{1-2\beta x}$$

$$+\frac{\gamma^2(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha)}{1-2\gamma x}.$$

Expanding $1/(1-2\alpha x)$, ..., into series, we obtain as the coefficient of $2^r x^r$:

$$C_r=\alpha^{r+2}(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha)+\beta^{r+2}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2$$

$$+\gamma^{r+2}(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha).$$

In this the coefficient of α^{r+2} equals

$$3\beta^2\gamma^2-(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)+3\alpha^2\beta\gamma-\alpha\beta\gamma(\alpha+\beta+\gamma)+\alpha\beta^3+\alpha\gamma^3-\beta\gamma^3-\beta^3\gamma$$

$$=3\beta^2\gamma^2+\beta\gamma-\alpha^4-9\alpha-2,$$

in view of (22) and the consequent relations

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)=3,$$

$$\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)+3\alpha\beta\gamma=-5,$$

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=-1.$$

Hence, if we set $S_n=\alpha^n+\beta^n+\gamma^n$, we obtain

(23) $$C_r-S3_r-S_{r+1}-2S_{r+2}-9S_{r+3}-S_{r+6}.$$

Since α, β, γ are the roots of y^3-y^2+y+1, we have Newton's identity

(24) $$S_k-S_{k+1}+S_{k-2}+S_{k-3}=0 \quad (k \gtrless 3).$$

Hence

(25) $$S_k+2S_{k-3}+S_{k-4}=0.$$

Hence (23) becomes

(26) $$C_r=10S_r+6S_{r+1}-8S_{r+2}.$$

Now C_r denoted the coefficient of $2^r x^r$ in the expansion of $44/D$. Hence the coefficient of $2^{r+2}x^{r+2}$ in the expansion of (21) is

$$\tfrac{1}{44}(C_{r+2}+2C_{r+1}-3C_r)=\tfrac{1}{44}(36S_{r+2}+28S_{r+1}-12S_r),$$

the last reduction following from (24) and (26). *Hence the coefficient of* x^{r+2} *in the series* (20) *is* $\tfrac{1}{11}(9S_{r+2}+7S_{r+1}-3S_r)2^{r+2}$.

By way of check, we note that $S_1=1$, $S_2=-1$, $S_3=-5$, as shown above. Hence $S_4=-5$, $S_5=1$, by (24). The coefficients of x^3 and x^4 are therefore

$$\tfrac{1}{11}(-45-7-3)2^3=-40,\ \tfrac{1}{11}(-45-35+3)2^4=-112.$$

Applying Waring's formula (Serret, *Algèbre*, I, page 449) to the cubic $y^3-y^2+y+1=0$, we obtain the explicit result

$$(27)\quad S_n=\Sigma\frac{(-1)^{\lambda_2+\lambda_3}n(\lambda_1+\lambda_2+\lambda_3)!}{(\lambda_1+1)!\,(\lambda_2+1)!\,(\lambda_3+1)!},\quad \begin{pmatrix}\text{summed for all integers} \geqq 0\\ \text{such that } \lambda_1+2\lambda_2+3\lambda_3=n\end{pmatrix}.$$

Second method. Equate expression (21) to $a_0+2a_1x+2^2a_2x^2+\dots+2^ra_rx^r+\dots$, clear of fractions, and compare coefficients. Then

$$1=a_0,\ 2=-a_0+a_1,\ -3=a_0-a_1+a_2,\ 0=a_0+a_1-a_2+a_3,\ \dots,$$

$$0=a_r+a_{r+1}-a_{r+2}+a_{r+3},\quad (r=0,\ 1,\ 2,\ \dots).$$

Noting that the determinant of the coefficients of the first n equations is 1, we get

$$(28)\qquad a_n=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1\\ -1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 2\\ 1 & -1 & 1 & 0 & 0 & 0 & \dots & 0 & -3\\ 1 & 1 & -1 & 1 & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 1 & -1 & 1 & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & \dots & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0\\ 0 & \dots & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0\\ 0 & \dots & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{vmatrix}$$

Expanding according to the elements of the last column, we get

$$(29)\qquad a_n=(-1)^n(D_n-2D_{n-1}-3D_{n-2}),$$

where D_n denotes the minor obtained upon deleting the first row and first column of (28). Adding the first row to the second in D_n, we get

$$D_n=\begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0\ldots 0 \\ 0 & 0 & 1 & 0 & 0 & 0\ldots 0 \\ 1 & 1 & -1 & 1 & 0 & 0\ldots 0 \\ 0 & 1 & 1 & -1 & 1 & 0\ldots 0 \\ 0 & 0 & 1 & 1 & -1 & 1\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}=-\begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0\ldots \\ 1 & 1 & 1 & 0 & 0 & 0\ldots \\ 0 & 1 & -1 & 1 & 0 & 0\ldots \\ 0 & 0 & 1 & -1 & 1 & 0\ldots \\ 0 & 0 & 1 & 1 & -1 & 1\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$=+\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0\ldots \\ 1 & -1 & 1 & 0 & 0 & 0\ldots \\ 0 & 1 & -1 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & -1 & 1 & 0\ldots \\ 0 & 0 & 1 & 1 & -1 & 1\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where the second determinant was reduced by adding its first row to its second. Expanding the third determinant according to its first column,

$$D_n=2D_{n-3}-D_{n-4}. \tag{30}$$

For $d_r=(-1)^rD_r$, relations (29) and (30) become

$$a_n=d_n+2d_{n-1}-3d_{n-2},\ d_n=-2d_{n-3}-d_{n-4}. \tag{31}$$

The second recursion relation is precisely recursion relation (25).

Third method. The existence of the three-term recursion relation, just mentioned, suggested the following method, which aside from its artificiality is perhaps the most satisfactory one of the three. We employ the identity

$$(1-2x+4x^2+8x^3)(1+2x)=1+16x^3+16x^4.$$

Let ${}_rC_k$ denote the number of combinations of r things k at a time. Then

$$\frac{1}{1+16x^3+16x^4}=\sum_{r=0}^{\infty}(-1)^r\sum_{k=0}^{r}{}_rC_k\,(16x^3)^k(16x^4)^{r-k}=\sum_{r,k}(-1)^r\,{}_rC_k\,16^rx^{4r-k}.$$

The coefficient of x^n is therefore

$$K_n=\sum_r(-1)^r\,{}_rC_{4r-n}16^r=\sum_{r=0}^{n'}(-1)^r16^r\,{}_rC_{n-3r}, \tag{32}$$

where n' is the greatest integer in $n/3$. Now expression (21) equals

$$\frac{(1+4x-12x^2)\,(1+2x)}{(1-2x+4x^2+8x^3)\,(1+2x)}=\frac{1+6x-4x^2-24x^3}{1+16x^3+16x^4}.$$

Hence the coefficients of x^n in series (20) is $K_n+6K_{n-1}-4K_{n-2}-24K_{n-3}$.

A SIMPLE EXISTENCE-PROOF FOR LOGARITHMS.

By JOHN WESLEY YOUNG, Instructor in Mathematics, Northwestern University.

Professor Osgood in an introduction to a recent paper by Mr. Bradshaw* says: "The student of mathematics and physics meets logarithms for the first time at an early stage. He is told that 'the logarithm of a number is the exponent of the power to which a certain number, taken as the base, must be raised in order to equal the given number.' The definition is purely formal. Probably the beginner has never seen a proof for the existence even of the fifth root of 2, ..., and if he has, it is not likely that it has meant anything to him. He tacitly assumes that every number has a positive qth root, q being any positive integer. It is then an easy step to any rational power, and the irrational powers are thought of as limiting cases, the principle being that, whenever one wants a limit in mathematics, the limit exists.

"Now all of these assumptions have been justified by rigorous ε-proofs in well known treatises on modern analysis. But the general student of mathematics and physics does not read these proofs, for they are uninteresting to him; and thus the great majority of students of the Calculus never see a proof that there is such a thing as a logarithm."

Mr. Bradshaw, in the paper cited, gives a proof by means of elementary calculus. In the following we give a proof which, while rigorous, makes use of only elementary algebra and which is much shorter than the proofs usually given.†

But first let us state precisely what is required, from an arithmetic point of view, to prove, the existence of such numbers as rational and irrational powers, logarithms, etc. Cantor's definition of an irrational number is as follows.‡ Suppose we have a series of rational numbers

$$\phi_0,\ \phi_1,\ \phi_2,\ \ldots\ldots,\ \phi_n,\ \ldots\ldots \tag{1}$$

with the property that for every number $\varepsilon>0$ we may find a number μ such that the relation

$$|\ \phi_{n+r}-\phi_n\ |\ <\epsilon$$

is satisfied, provided only that n be greater than μ, r being any positive integer; then if $\underset{n\doteq\infty}{\text{Lim}}\ [\phi_n]$ is not a rational number, the series (1) *defines* an irrational number equal to $\underset{n\doteq\infty}{\text{Lim}}[\phi_n]$. In order to prove that a certain irrational number a *exists* it is necessary to establish a series (1) such that $|\ a-\phi_n\ |<\epsilon$, provided only that $n>\mu$.

*J. W. Bradshaw, The Logarithm as a Direct Function; *Annals of Mathematics* (January, 1903), 2nd series, v. 4, p. 51.

†Stolz, in his *Allgemeine Arithmetik*, devotes 24 pages to the discussion of powers, roots and logarithms; cf. Vol. I, Chap. 8, pp. 125-148.

‡Stolz, *Allgemeine Arithmetik*, I, p. 105.

We now turn to the problem of proving the existence of a logarithm of any positive number. Of irrational operations we shall assume only the possibility of extracting the square root of any positive number; that such a root exists may be seen geometrically or (better) the ordinary ϵ-proof may be supplied, which in this case is very simple.* Consider two sequences of numbers G_0 and A_0, the former in geometric, the latter in arithmetic progression:

$$(G_0) \quad \ldots\ldots, a^{-m}, \ldots\ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots\ldots, a^n, \ldots\ldots$$

$$(A_1) \quad \ldots\ldots, -mb, \ldots\ldots, -2b, -b, 0, b, 2b, \ldots\ldots, nb, \ldots\ldots$$

where a, b are any positive numbers ($a \neq 1$). Establish a one-to-one correspondence between the terms of the sequences, such that 1 of G_0 corresponds to 0 of A_0, a of G_0 to b of A_0, and in general a^k of G_0 to kb of A_0, for all integral, positive or negative, values ~~of k~~. Then if p, q be any two numbers of G_0, and if $\phi(p)$, $\phi(q)$ denote the corresponding numbers of A_0, we have

$$\phi(pq) = \phi(p) + \phi(q). \tag{2}$$

It is our object to prove the existence of numbers $\phi(x)$, which have the property (2), for all real positive values of x. To do this insert a geometric mean between some pair of consecutive numbers of G_0, and corresponding thereto insert an arithmetic mean between the corresponding pair of A_0, and suppose this done for every pair of consecutive numbers of the two sequences. We thus obtain two new corresponding sequences G_1, A_1, such that if now p, q be any two numbers of G_1 and $\phi(p)$, $\phi(q)$ the corresponding numbers of A_1, relation (2) still holds. By treating G_1, A_1 as we did G_0, A_0, we obtain two new sequences G_2, A_2; and this process may be continued *ad libitum*. After n steps we have two sequences G_n, A_n, of which the first has a common ratio equal to $\sqrt[2^n]{a}$ and the second a common difference b/n. The latter approaches 0 as a limit when n increases indefinitely; whereas the former with increasing n must approach the limit 1. We have then

$$\sqrt[2^n]{a} = 1 + \varepsilon_n,$$

where ε_n may be made smaller than any assigned positive number by taking n sufficiently large.

Now let x be any (finite) positive real number whatever. Either x is contained in some G_n, and then $\phi(x)$ is already defined as the corresponding number of A_n; or x is not contained in any G_n no matter how large we take n. In the latter case suppose x lies between the two consecutive numbers x_n and $x_n(1+\varepsilon_n)$ of G_n; then we have

$$x - x_n < x_n \varepsilon_n.$$

*Cf. e. g. Fischer and Schwatt, *Higher Algebra*, p. 267, or Stolz, *loc. cit.* p. 129.

Since ϵ_n can be made as small as desired by taking n sufficiently large, it follows that as n increases indefinitely the sequence

$$x_0, x_1, x_2, \ldots\ldots, x_n$$

approaches x as a limit. From a fundamental theorem on limits it follows that the corresponding sequence

$$\phi(x_0), \phi(x_1), \phi(x_2), \ldots\ldots, \phi(x_n)$$

approaches a limit, which we *define* as $\phi(x)$.

The function $\phi(x)$, $=y$ say, is now defined for all values of x such that $0 < x < +\infty$. Moreover y, $=\phi(x)$, *is clearly a one-valued and continuous function in this iuterval, satisfying the functional relation* (2).

Similar reasoning in the opposite direction (*i. e.* from the arithmetical progression to the geometric) shows that *the inverse function* $x=\phi^{-1}(y)=\psi(y)$, *say, is one-valued and continuous in the interval* $-\infty < y < +\infty$ *and satisfies the functional relation* $\psi(\mu)\psi(\nu)=\psi(\mu+\nu)$.

We must now prove that $\phi(x)$ is identical with $\log_k(x)$, or, what is the same, that $\psi(y)$ is identical with k^y, where the base k is $\phi(1)$.*

From (2) we obtain readily

$$\phi(x^m)=m\phi(x) \tag{3}$$

when m is any positive integer. From this we may show that *every positive number a has just one mth root*. For, if $\phi(a)=b$, the number $x=\psi(b/m)$ is easily shown to satisfy the relation $x^m=a$, and conversely the only positive number x satisfying this relation is $\psi(b/m)$.†

Further, if the *meaning* of a^α, where α is any rational number (positive, negative, or zero), be defined as usual, this proves the *existence* of all rational powers of a.

Now, (3) *holds for all rational values of m*. For, replacing x by $x^{1/t}$ and m by t in (3) we obtain

$$\phi(x^{1/t})=(1/t)\phi(t)$$

and thence easily

$$\phi(x^{s/t})=(s/t)\phi(x),$$

s, t being integers, and s positive, which is no restriction. By replacing x and m by $k=\psi$ (1) and y respectively in (3), we obtain

$$\phi(k^y)=y\phi(k)=y, \text{ whence } k^y=\psi(y)$$

for all rational values of y. If the sequence of rational numbers

*Cf. in this connection, Tannery, *Theorie des functions d'une variable*, p. 120, §82.
*Bradshaw, *loc. cit.*, page 58.

$$y_0, y_1, y_2, \ldots, y_n$$

approaches a limit y, when n increases indefinitely, the sequence

$$\psi(y_0), \psi(y_1), \psi(y_2), \ldots, \psi(y_n)$$

approaches the limit $\psi(y)$, and hence *the identical sequence*

$$k^{y_0}, k^{y_1}, k^{y_2}, \ldots k^{y_n}$$

*approaches a limit.** This limit is *defined* as k^y.† Hence for all real values of y

$$\psi(y)=k^y \text{ and hence } \phi(x)=\log_k(x).$$

EVANSTON, ILLINOIS, *September 21, 1903.*

CONVERSE AND OPPOSITE PROPOSITIONS.

By C. M. HIMEL, Baker-Himel School, Knoxville, Tenn.

Wentworth, in the revised edition of his Plane Geometry, page 5, makes the following confusing statements:

"If a direct proposition and its opposite are true, the converse proposition is true; and if a direct proposition and its converse are true, the opposite proposition is true."

"Thus, if it were true that

1. If an animal is a horse, the animal is a quadruped;
2. If an animal is not a horse, the animal is not a quadruped;

it would follow that

3. If an animal is a quadruped, the animal is a horse.

Moreover, if 1 and 3 are true, then 2 would be true."

The statements should read: *Whether a direct proposition is true or not, if the converse is true the opposite is true; if the opposite is true the converse is true.*

Thus, in the above example, if 1 be true, then 3 would follow; if 3 be true, then 2 would be true. Consider the general case:

1. Proposition: If a is b, then c is d.
2. Converse: If c is d, then a is b.
3. Opposite: If a is not b, then c is not d.

We may prove that if either of the last two, irrespective of the first, is true, the other is also true.

First. Suppose you know that the converse is true, and you wish to prove

*This required proof.

†Cf. Stolz, *loc. cit.*, p. 138; Tannery, *Theorie des fonctions d'une variable*, p. 114.

the opposite. You are given that a is not b, and you wish to prove that c is not d. It is not, because, if it is, then a is b; but a is not b, therefore c is not d.

Second. By a similar proof, if the opposite be true, the converse is true.

For example, from the theorem [compare the proof in Wentworth's Revised Geometry, p. 70, Ex. 55]: If two angles of a triangle are not equal, their bisectors are not equal, one may conclude: If the bisectors of two angles of a triangle are equal, the triangle is isosceles. The proof of the first theorem is, indeed, a proof of the second.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

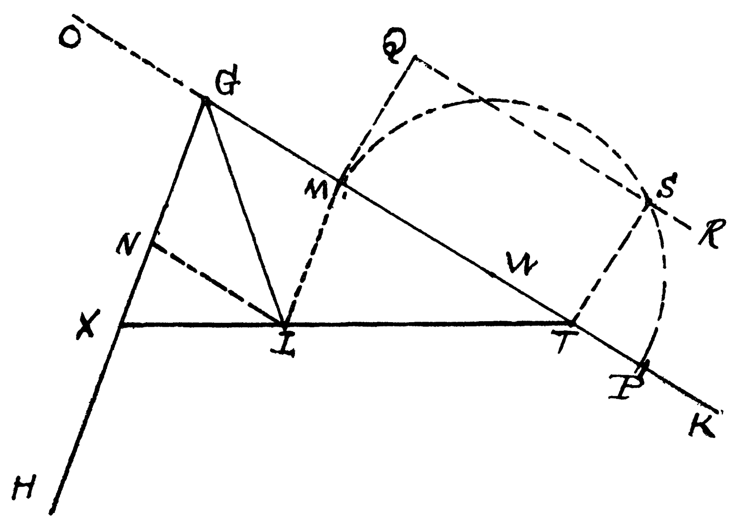
203. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud; England.

Show that two parabolae can always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex is parallel to the opposite side.

No solution has been received.

204. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides. (Phillips and Fisher).



Solution by G. I. HOPKINS, J. SCHEFFER, and G. B. M. ZERR.

Let AB be the sum of the two sides, CD the bisector, and F the given angle. Make $\angle HGK = \angle F$ and bisector $GL=CD$. Draw LM parallel to HG and NL parallel to GK. Extend GK making $GO=NG$. Make $OP=AB$, and on MP draw the semicircle MSP. Draw the perpendicular $MQ=ML$, also QR parallel to MP and ST perpendicular to MP. Through L draw TX, then TGX is the required triangle; for $TS^2=MT.PT=QM^2=ML^2$. From similar triangles XNL and LMT, $XN:LM::NL:MT$. Since $MLNG$ is a rhombus $LM=NL$. $\therefore XN.MT=ML^2$. $\therefore MT.TP=MT.XN$, whence $NX=TP$. $\therefore GX+GP=OP=AB$.

CALCULUS.

168 Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

The tangent of what Cartesian curve makes an x-intercept always m times as long as the corresponding y-intercept.

Solution by J. SCHEFFER.

Let the equation of the tangent be $y-y'=\frac{dy'}{dx'}(x-x')$. Consequently the x-intercept $=x'-y'\frac{dx'}{dy'}$, and the y-intercept $=y'-x'\frac{dy'}{dx'}$; therefore, omitting the accents, by the condition imposed upon the problem

$$-\frac{y}{p}+x=m(y-px)\,;\ \left(p=\frac{dy}{dx}\right),$$

whence $mxp^2-(my-x)p-y=0$; or, arranged differently, $(px-y)(mp+1)=0$, whence $px-y=0$ and $mp+1=0$. From the former of these two equations we get $y=ax$; and from the second $my+x=b$, where a and b are arbitrary constants. Both equations represent straight lines, the first one of which passes through the origin.

MECHANICS.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

Required the time for a tree, considered as a material line of uniform density, length $a=100$ feet, to fall; the tree being inclined $\phi=1'$ from perpendicular.

Solution by the PROPOSER.

From Mechanics we find that

$$\frac{\sqrt{(2gh)}t}{l}=\int\frac{d\theta}{\sqrt{[1-(2l/h)\sin^2\frac{1}{2}\theta]}}\ \ldots\ldots(1),$$

where g=gravity=32, l=length to center of oscillation=$\frac{2}{3}a$, h=height of zero velocity=$2l$ without a sensible error, t=time, θ=angle counted from the down point. Reducing, we have,

$$t=\tfrac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\int_{\frac{1}{2}\pi}^{\pi-\phi}\frac{d\theta}{\cos\frac{1}{2}\theta}=\sqrt{\left(\frac{l}{g}\right)}\Big[\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{4}\theta)\Big]_{\frac{1}{2}\pi}^{\pi-\phi}$$

$$=\sqrt{\left(\frac{l}{g}\right)}\log\frac{\cot\frac{1}{4}\phi}{\tan\frac{3}{8}\pi}=\sqrt{\left(\frac{l}{g}\right)}(8.647491).$$

But $l=\frac{2}{3}$ of 100. $\therefore t=\sqrt{(\frac{100}{48})}(8.647491)=12.481$ seconds.

AVERAGE AND PROBABILITY.

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139. Proposed by **L. C. WALKER, A.M.**, Professor of Mathematics, Colorado School of Mines, Golden, Col.

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

No solution has been received.

140. Proposed by **L. C. WALKER, A.M.**, Professor of Mathematics, Colorado School of Mines, Golden, Col.

Obtain the average area of a triangle formed by a tangent to the four-cusped hypocycloid and the co-ordinate axes.

Solution by **J. E. SANDERS**, Hackney, Ohio.

The curve's equation is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. The length of the perpendicular from the origin on the tangent is

$$\frac{xdx-ydy}{\sqrt{(dx^2+dy^2)}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{\sqrt{(y^{-\frac{2}{3}}+x^{-\frac{2}{3}})}}=\sqrt[3]{(axy)}.$$

The equation to the tangent is $x'/x^{\frac{1}{3}}+y'/y^{\frac{1}{3}}=a^{\frac{2}{3}}$, and its length between the axis$=a^{\frac{2}{3}}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}=a$.

$\therefore$ The area of the triangle is $A=\frac{1}{2}a\sqrt[3]{(axy)}=\frac{1}{2}a^2\sin\theta\cos\theta$, if $x=a\cos^3\theta$, and $y=a\sin^3\theta$.

Then, if the tangent is through any point on the curve, the average area is

$$\triangle=\frac{1}{s}\int Ads.$$

But $ds=a^{\frac{1}{3}}x^{-\frac{1}{3}}dx=3a\sin\theta\cos\theta\,d\theta$ and $s=\frac{3}{2}a$.

$\therefore\triangle=a^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\cos^2\theta\,d\theta=\frac{1}{16}\pi a^2$, or, if the tangent is to be a random line, the average area is

$$\triangle_1 = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \tfrac{1}{2}a^2 \sin\theta \cos\theta \, d\theta = \frac{a^2}{2\pi}.$$

As the problem is stated it seems that either of the above suppositions are allowable.

Also solved by the *PROPOSER*.

142. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., Washington, D. C.

Two points are taken at random in the arc of a semi-circle, and a third point anywhere in its base. Find the probability that the triangle formed by them is acute.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let A, B be the random points on the arc of the semi-circle, MN its diameter or base, C its center. On AB as diameter describe a circle with center F. At A, B, perpendicular to AB, draw AD, BE meeting MN in D and E. From A draw AL parallel to MN meeting BE in L, and from F draw FG perpendicular to MN meeting MN in G. If the circle on AB does not intersect MN. the third point can be anywhere on DE; if the circle on AB intersects MN, then the third point can be anywhere on DE without this circle.

$\therefore$ If the circle on AB intersects MN in H, then the third point K can be anywhere on DH or EK. Let $CA=CB=r$, $\angle MCF=\theta$, $\angle ACF=\varphi$. Then $AB=2r\sin\varphi$, $AL=DE=2r\sin\varphi \operatorname{cosec}\theta=Q$, $FC=r\cos\varphi$, $FG=r\cos\varphi \sin\theta$. When $FG=FB$ we get $r\sin\varphi=r\cos\varphi\sin\theta$.

$\therefore \tan\varphi=\sin\theta$, $HG=\sqrt{(HF^2-FG^2)}=\sqrt{(AF^2-FG^2)}$.

$\therefore HG=r\sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}$.

$\therefore DH+EK=DE-2HG=2r[\sin\varphi \operatorname{cosec}\theta-\sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}]=Q'$.

Let $\varphi=\tan^{-1}(\sin\theta)=\varphi'$, p=required chance.

$$\therefore p=\frac{\left[\frac{1}{2r}\int_0^{\frac{1}{2}\pi}\int_0^{\phi} Q\,d\theta\,d\varphi+\frac{1}{2r}\int_0^{\frac{1}{2}\pi}\int_{\phi'}^{\theta} Q'\,d\theta\,d\varphi\right]}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta} d\theta\,d\varphi}$$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\left[\int_0^{\phi'} \sin\varphi \operatorname{cosec}\theta\,d\varphi+\int_{\phi'}^{\theta}(\sin\varphi \operatorname{cosec}\theta\right.$$

$$\left.-\sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}d\varphi\right]d\theta=\frac{4}{\pi^2}+\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\phi'}^{\theta}\frac{\tan^2\varphi\sec\varphi\,d\theta\,d\varphi}{\sqrt{(\tan^2\varphi-\sin^2\theta)}}.$$

The last term can be expanded and then integrated in series.

143. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

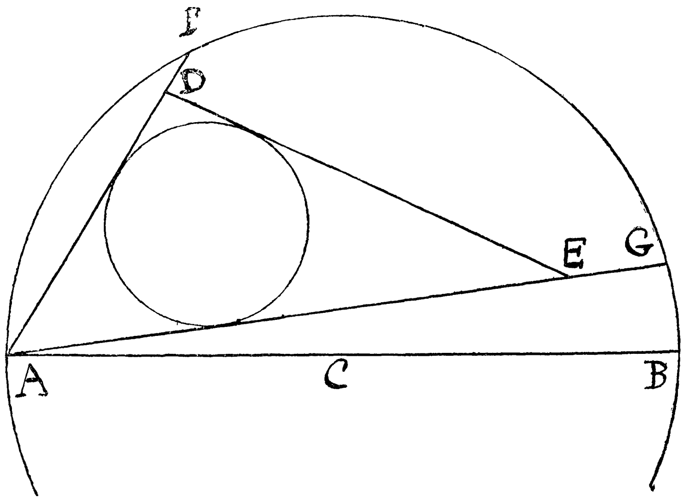
The extremities of two equal lines drrwn from a fixed point in the circumference of a given circle is joined. Find the average area of the circle inscribed in the triangle formed.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let C be the center of the given circle, A the given point on the circumference, $AC=r$, $AD=AE=x$, $\angle FAB=\theta$, $\angle FAG=2\phi$. Then $AF=2r\cos\theta=x'$, $DE=2x\sin\phi$, perimeter $ADE=2x(1+\sin\phi)$.

Area $ADE=\frac{1}{2}x^2\sin2\phi=x^2\sin\phi\cos\phi$. Area of inscribed circle$=u$

$$=\frac{\pi(\text{area}ADE)^2}{(\frac{1}{2}\text{perimeter}ADE)^2}=\frac{\pi x^2\sin^2\phi\cos^2\phi}{(1+\sin\phi)^2}.$$



The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of x, 0 and x'.

$$\therefore \Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}u\,d\theta\,d\phi\,dx}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}d\theta\,d\phi\,dx}=\frac{\pi}{r(\pi-2)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}\frac{\sin^2\phi\cos^2\phi}{(1+\sin\phi)^2}d\theta\,d\phi\,x^2dx$$

$$=\frac{8\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\frac{\cos^3\theta\sin^2\phi(1-\sin\phi)}{(1+\sin\phi)}d\theta\,d\phi$$

$$=\frac{4\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\left(12+\sin\theta\cos\theta-4\cos\theta-5\theta-\frac{8}{1+\tan\frac{1}{2}\theta}\right)\cos^3\theta\,d\theta$$

$$=\frac{4\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\left(8+\sin\theta\cos\theta-4\cos\theta-5\theta-\frac{4\cos\theta}{1+\sin\theta}\right)\cos^3\theta\,d\theta$$

$$=\frac{\pi r^2(1936-615\pi)}{135(\pi-2)}.$$

144. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

In a circular park 400 feet in diameter are four *equal* circular ponds of *variable* diameter. What is the probability that a sightless person walking in a straight line from the center of the park will step into a pond?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

In this solution we shall assume that a random radius (from center C of circular park of radius r) does not intersect more than one pond on the same side of the center. The required chance, p, then is four times the chance that a random radius will intersect a pond lying wholly upon the surface of a quadrant of the given circle.

Let P be center of pond, x its radius, θ=angle between tangents from C to pond, φ=angle at P between radii to points of contact. Let $CP=y$. Then $\theta=2\sin^{-1}(x/y)$, $\varphi=\frac{1}{2}\pi-2\sin^{-1}(x/y)$. The limits of x are 0 and $r(\sqrt{2}-1)=x'$; of y, $x\sqrt{2}$ and $r-x$.

$$\therefore p=4\frac{\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\theta\varphi\, dxdy}{\int_0^{x'}\int_{x\sqrt{2}}^{r-x}2\pi\varphi\, dxdy}. \quad 2\pi\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\varphi dxydx$$

$$=\frac{\pi}{2}\int^{x'}\left[\pi(r-x)^2-4(r-x)^2\sin^{-1}\left(\frac{x}{r-x}\right)-4x\sqrt{(r^2-2rx)}+4x^2\right]dx$$

$$=\frac{\pi r^3}{18}(3\pi-28+16\sqrt{2}).$$

$$\therefore p=\frac{72}{\pi r^3(3\pi-28+16\sqrt{2})}\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\theta\varphi\, dxdy$$

$$=\frac{36}{\pi r^3(3\pi-28+16\sqrt{2})}\int_0^{x'}\left[\pi(r-x)^2\sin^{-1}\left(\frac{x}{r-x}\right)+\pi x^2+\pi x\sqrt{(r^2-2rx)}\right.$$

$$\left.-4(r-x)^2\left\{\sin^{-1}\left(\frac{x}{r-x}\right)\right\}^2-9x\sqrt{(r^2-2rx)}\sin^{-1}\left(\frac{x}{r-x}\right)-8x^2\log\left(\frac{r-x}{x\sqrt{2}}\right)\right]dx$$

$$=\frac{4}{(3\pi-28+16\sqrt{2})}\left[4\sqrt{2}-3+\frac{8}{\pi}-\frac{8\sqrt{2}}{\pi}+\frac{8}{\pi}\log(2-\sqrt{2})\right].$$

MISCELLANEOUS.

138. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find an invariant of the *third degree* in the coefficients of a ternary cubic.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $U=ax^4+by^4+cz^4+4(a_2x^3y+a_3x^3z+b'_3y^3z+b_1y^3x+c_1z^3x+z_3z^3y)$

$$+6(dy^2z^2+ez^2x^2+fx^2y^2)+12xyz(lx+my+nz)$$

be the ternary quartic. The required invariant written out in full is

$$\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^4}-4\left(\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dy}+\frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx^3\,dz}+\frac{d^4u}{dz^4}\cdot\frac{d^4u}{dx^3\,dy}\cdot\frac{d^4u}{dy^3dx}\right)$$

$$+3\left[\frac{d^4u}{dx^4}\left(\frac{d^4u}{dy^2\,dz^2}\right)^2+\frac{d^4u}{dy^4}\left(\frac{d^4u}{dx^2\,dz^2}\right)^2+\frac{d^4u}{dz^4}\left(\frac{du^4}{dx^2\,dy^2}\right)\right]$$

$$+4\left(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dx}+\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dy}\right)-12\left(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dxdydz^2}\cdot\frac{d^4u}{dy^2\,dz^2}\right.$$

$$+\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dy^2\,dz^2}+\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dz^2\,dx^2}+\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dz^2\,dx^2}$$

$$\left.+\frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx^2\,dy^2}+\frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx^2\,dy^2}\right)+12\left(\frac{d^4u}{dx^2dydz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dx}\right.$$

$$\left.+\frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^3dy}+\frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3\,dz}\right)+12\left[\frac{d^4u}{dy^2\,dz^2}\cdot\left(\frac{d^4u}{dx^2\,dy\,dz}\right)^2\right.$$

$$\left.+\frac{d^4u}{dz^2\,dx^2}\cdot\left(\frac{d^4u}{dx\,dy^2\,dz}\right)^2+\frac{d^4u}{dx^2\,dy^2}\cdot\left(\frac{d^4u}{dx\,dy\,dz^2}\right)^2\right]+6\frac{d^4u}{dy^2\,dz^2}\cdot\frac{d^4u}{dy^2\,dx^2}\cdot\frac{d^4u}{dx^2\,dz^2}$$

$$-12\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx\,dy\,dz^2}.$$

$\therefore abc-4(ab_3c_2+bc_1a_3+ca_2b_1)+3(ad^2+be^2+cf^2)+4(a_2b_3c_1+a_3b_1c_2)$ $-12(a_2nd+a_3md+b_1ne+b_3le+c_1mf+c_2lf)+12(lb_1c_1+mc_2a_2+na_3b_3)+12(dl^2$ $+em^2+fn^2)+6def-12lmn.$

139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Given the roots of a binary cubic, to find the roots of its two inbependent covariants.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $ax^3+3bx^2y+3cxy^2+dy^3=0$ be the cubic. The two independent covariants are,

$$(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2=0. \qquad (1)$$

$$(ad^2-3bcd+2c^3)x^3+3(2b^2d-acd-bc^2)x^2y+3(2ac^2-abd-b^2c)xy^2$$
$$+(a^2d-3abc+2b^3)y^3=0. \qquad (2)$$

Let $x/y=z$, then $az^3+3bz^2+3cz+d=0$ is the cubic. Let α, β, γ be the gamma roots of this cubic. Then

$$A=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$=-(18/a^2)(ac-b^2);$$
$$B=2[\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2]=-(18/a^2)(ad-bc);$$
$$C=\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2=-(18/a^2)(bd-c^2).$$

$\therefore$ The quadratic covariant is $Az^2+Bz+C=0$. The roots of this equation are $[B\pm\sqrt{(B^2-4AC)}]/2A$. Substituting the values of A, B, C in terms of α, β, γ, the roots are

$$\frac{\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\pm(\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)\sqrt{(-3)}}{(\alpha-\beta)^2+(\gamma-\alpha)^2+(\beta-\gamma)^2}.$$

$$D=2\beta\gamma-\alpha\beta-\alpha\gamma=3\beta\gamma-(\beta\gamma+\alpha\beta+\alpha\gamma)=3\beta\gamma-3c/a;$$
$$E=2\gamma\alpha-\beta\gamma-\beta\alpha=3\gamma\alpha-3c/a;$$
$$F=2\alpha\beta-\gamma\alpha-\gamma\beta=3\alpha\beta-3c/a;$$
$$G=\alpha+\beta-2\gamma=\alpha+\beta+\gamma-3\gamma=-3b/a-3\gamma;$$
$$H=\beta+\gamma-2\alpha=-3b/a-3\alpha;$$
$$K=\gamma+\alpha-2\beta=-3b/a-3\beta.$$

$$\therefore DEF=27(\alpha\beta-c/a)(\gamma\alpha-c/a)(\beta\gamma-c/a)=(27/a^3)(ad^2-3bcd+2c^3);$$
$$GHK=-27(\alpha+b/a)(\beta+b/a)(\gamma+b/a)=(27/a^3)(a^2d-3abc+2b^3);$$
$$DEG+DFK+EFG=-27[(\beta\gamma-c/a)(\gamma\alpha-c/a)(\gamma+b/a)+(\beta\gamma-c/a)$$
$$\times(\alpha\beta-c/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha\beta-c/a)(\alpha+b/a)]$$
$$=-(81/a^3)(2b^2d-acd-bc^2);$$

$$DGK+EGH+FHz=27[(\beta\gamma-c/a)(\gamma+b/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha+b/a)$$
$$\times(\gamma+b/a)+(\alpha\beta-c/a)(\alpha+b/a)(\beta+b/a)]$$
$$=(81/a^3)(2ac^2-abd-b^2c).$$

$\therefore$ The cubic covariant is

$$DEFz^3+(DEG+DFK+EFH)z^2+(DGK+EGH+FHK)z+GHK=0.$$

$\therefore z^3+(G/F+K/E+H/D)z^2+(GK/EF+GH/DF+HK)DE)z+GHK/DEF=0.$

$\therefore (z+G/F)(z+K/E)(z+H/D)=0.$

$\therefore$The roots are $-G/F$, $-K/E$, $-H/D$, or

$$\frac{2\gamma-\alpha-\beta}{2\alpha\beta-\gamma\alpha-\gamma\beta},\quad \frac{2\beta-\alpha-\gamma}{2\gamma\alpha-\beta\gamma-\beta\alpha},\quad \frac{2\alpha-\beta-\gamma}{2\beta\gamma-\alpha\beta-\alpha\gamma}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

185. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Without introducing radicals, eliminate x and y from the equations

$$ax^2+bx+c=0,\ ay^2+by+d=0,\ ax^2y^2+bxy+e=0.$$

186. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Eliminate x and y from the equations

$$ax^3+bx^2+cx+d=0,$$
$$ay^3+by^2+cy+e=0,$$
$$ax^3y^3+bx^2y^2+cxy+f=0,$$

the eliminant to be rational in d, e, f.

GEOMETRY.

207. Proposed by W. W. HART, University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into n equal parts by ruler and compass when and only when n is a prime of the form $2^{2^p}+1$.

The following construction gives good partial results for n equals *any* integer. If AB is the diameter of the circle, and C is the vertex of the equilateral triangle ABC, and if D is a point on AB at the distance $2AB/n$ from A, then draw the line CD cutting the circle at E and F; E being the more remote from C. $AE=1/n$ circumference approximately. For low values of n this method is very practical; is it practical in general? How great is the error?

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Tangents drawn to two confocal parabolae from a point on the common tangent intersect at the same angle as the axes of the parabolae.

MECHANICS.

163. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

A particle A, mass m, rests on a smooth horizontal plane and is attached by two inelastic strings to masses m_1, m_2 at points B and C such that BAC is a right angle. If a blow is given A at an angle θ to AB, find the initial direction of motion of m, and equations for initial motion of the particles m_1 and m_2.

NOTES.

The students of The University of Chicago have organized an Elementary Mathematics Club.

Professor J. B. Shaw of Kenyon College, has accepted a call to Millikan University at Decatur, Ill.

Mr. F. E. Knowles has been appointed to an instructorship in mathematics at the University of Oklahoma.

Mr. Floid Field has been appointed Instructor in Mathematics at the Academy of Northwestern University.

Mr. W. O. Beal has been appointed to an instructorship in mathematics at the Illinois College, Jacksonville, Ill.

Mr. J. W. Brister has been elected Associate Professor of Mathematics at the University of Nashville, Nashville, Tenn.

Dr. C. M. Mason (Göttingen) has been appointed Instructor of Mathematics at the Massachusetts Institute of Technology.

Professor H. B. Leonard of the Illinois Wesleyan University will spend this year in doing graduate work at The University of Chicago.

Professor G. B. M. Zerr is superintending the erection of a large electro-chemical plant for the manufacture of bleach at Parsons, W. Va.

Mr. A. B. Pierce, formerly Instructor in Mathematics at the University of California, took his Doctor's degree at the University of Zürich, Switzerland, this summer.

ERRATA IN THIS NUMBER.

Page 219, middle, *for* al *read* all; page 220, line 6, insert (
Page 221, in (10), *read* $(2-b)t^3$; page 222, line 12, *read* It remains...belongs
Page 223, line 11, *read* $c=\pm1$; line 17, *read* reduce to
Page 224, in (23), *read* $C_r=3S_r-$; *after* y^3-y^2+y+1 *put* $=0$; in (24), *for* S_{k+1} *read* S_{k-1}.

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CONCERNING SIMPLE CONTINUED FRACTIONS.

By PROFESSOR THOMAS E. McKINNEY, Marietta, Ohio.

1. In his *Lecons sur la théorie des fonctions*, Chapter II, Borel gives an account of Liouville's demonstration of the existence of non-algebraic numbers. As an application of the theorem on which this demonstration rests he considers the approximation of incommensurable numbers by the method of simple continued fractions and is led to an important proposition in the elementary theory of the subject. As his treatment of this proposition is different from that adopted in any standard text on Algebra—in fact the proposition in the general form is not is not in any Algebra with which I am acquainted—it is proposed to give with some modification of arrangement and detail Borel's account of the theorem and its application to continued fractions.

2. Denote by ξ a real number satisfying an irreducible equation $f(x)=0$ with integral coefficients, of degree n in x. Let p/q be a rational fraction in its lowest terms and let both p/q and ξ lie in a certain interval α, β, where α and β are both finite but otherwise arbitrary. Then the following proposition may be established:

THEOREM. *A positive number M can be determined such that whatever be the number p/q in the interval α, β the inequality*

$$(1) \quad \left|\frac{p}{q}-\xi\right|>\frac{1}{Mq^n}$$

is satisfied.

Since $f(x)$ is a polynomial in x it is finite and continuous in the interval

α, β and has a derivative which is also a polynomial and consequently finite in the same interval. Hence there is a positive number M such that

$$(2) \qquad |f'(x)| < M$$

in the entire interval α, β.

Inasmuch as $f(x)$ is a polynomial with integral coefficients,

$$f(p/q) = A/q^n,$$

where A is an integer. If p/q is not a root of the equation $f(x)=0$ then $|A| \geqq 1$, and consequently

$$(3) \qquad \left| f\left(\frac{p}{q}\right) \right| \geqq \frac{1}{q^n}.$$

Now by Taylor's formula, since $f(\xi)=0$,

$$f(p/q) = (p/q-\xi) f'[\xi+\theta(p/q-\xi)], \quad 0<\theta<1.$$

Since $\xi+\theta(p/q-\xi)$ lies in the interval α, β we may apply (2). Hence

$$|f(p/q)| < |p/q-\xi| M.$$

Then by inequality (3) we have (1) after a slight reduction.

3. Since ξ is a real number it may be expressed in the form of a continued fraction, in the usual notation

$$\xi = (a_0, a_1, \ldots, a_m, \xi_{m+1}),$$

where $\xi_{m+1} \geqq 1$, while $a_1, a_2, \ldots, a_m$ are positive integers, not zero. Denote the $(i+1)$th convergent of this continued fraction by p_i/q_i. Then

$$\xi = \frac{p_m \xi_{m+1} + p_{m-1}}{q_m \xi_{m+1} + q_{m-1}}$$

and

$$\frac{p_m}{q^m} - \xi = -\frac{p_{m-1}\, q_m - p_m q_{m-1}}{q_m^{\,2}\left(\xi_{m+1} + \dfrac{q_{m-1}}{q_m}\right)} = \frac{(-1)^{m+1}}{q_m^{\,2}\left(\xi_{m+1} + \dfrac{q_{m-1}}{q_m}\right)}.$$

Since $a_m > 0$, $q_{m-1}/q_m > 0$, and, consequently,

$$(4) \qquad \left| \frac{p_m}{q_m} - \xi \right| < \frac{1}{\xi_{m+1}\, q_m^{\,2}}.$$

Replacing p and q in inequality (1) by p_m and q_m respectively, and comparing the results with inequality (4) we have

$$(5)\quad \xi_{m+1} < M q_m^{\,n-2}.$$

Hence the inequality, as given by Borel,

$$(6)\quad a_{m+1} < M q_m^{\,n-2}.$$

4. When $n=1$, ξ is rational and inequality (6) takes the form

$$a_{m+1} < M/q_m.$$

Since q_m increases without limit with m, by taking m great enough a_{m+1} may be made less than any assigned positive number, however small. Now this involves a contradiction since always $a_{m+1} \geqq 1$. Hence the elementary

THEOREM. *A rational number is represented by a terminating continued fraction.*

5. When $n=2$, the inequality (6) becomes

$$(7)\quad a_{m+1} < M.$$

To determine M explicitly let

$$f(x) \equiv ax^2 + bx + c, \quad b^2 - 4ac \equiv D > 0,$$

and let ξ represent either of the values $(-b \pm \sqrt{D})/a$. Since

$$\left| \frac{p_m}{q_m} - \xi \right| < 1, \qquad m=0, 1, 2, \ldots\ldots$$

two numbers α, β can be chosen in the interval $\xi-1$, $\xi+1$ so that ξ and every convergent p_m/q_m, $m=0, 1, 2, \ldots\ldots$, shall lie in the interval α, β. Hence when for M the greater of the values $|f'(\xi-1)|$, $|f'(\xi+1)|$ is taken, then in the interval α, β, $|f(x)| < M$. We find that

$$M = 2(|a| + \sqrt{D}).$$

Comparing this with inequality (7) we have the following

THEOREM. *In the continued fraction representing either quadratic irrationality $(-b \pm \sqrt{D})/a$, a, b, D integers, $D>0$, every partial denominator after the first is less than* $2(|a| + \sqrt{D})$.

6. Denote by ε a small positive number. Then α and β may be chosen in the interval $\xi-\varepsilon$, $\xi+\epsilon$ so that for i great enough ξ and every convergent of order greater than i lies in the interval α, β. As in the preceding instance take for M the greater of the two numbers $|f'(\xi-\epsilon)|$, $|f'(\xi+\epsilon)|$. Then in the interval α, β,

$$|f'(x)| < M, \quad M = 2(|a|\varepsilon + \sqrt{D}).$$

Now let $2\sqrt{D}+d$ be the integer next greater than $2\sqrt{D}$ and choose ε so that

$$\varepsilon<\frac{d}{2\,|\,a\,|}, \text{ whence } M<2\sqrt{D}+d.$$

Hence by inequality (7), since a_{m+1} is an integer,

$$a_{m+1}<2\sqrt{D}$$

for every m, $m>i$. This is the well known

THEOREM. *In the continued fraction representing either quadratic irrationality* $(-b\pm\sqrt{D})/a$, *a, b, D integers, $D>0$, every partial denominator from one of a certain rank on, is less than* $2\sqrt{D}$.

7. The more general theorem of which this is a special case is the following

THEOREM. *In the continued fraction representing the real number ξ, where ξ is the root of an irreducible equation with integral coefficients $f(x)=0$ of degree n, every partial denominator a_{m+1}, from one of a certain rank on, satisfies the inequality* $a_{m+1}<\,|\,f'(\xi)\,|\,q_m{}^{n-2}$.

PROPERTIES OF THE FUNCTION $(1+a)^x$.

By ANTONIO LLANO, Scranton, Pa.

The following demonstrations of some well known theorems are submitted as being simpler and more systematic than those usually given. The binomial $1+a$ is supposed positive, or $a>-1$.

THEOREM I. *If $x>1$, then* $(1+a)^x>1+ax$.

Let $x=u/v$, where $u>v$, and put $1+a=z^v$. We have

$$\frac{z^u-1}{z^v-1}=\frac{z^{u-1}+z^{u-2}+\ldots\ldots+1}{z^{v-1}+z^{v-2}+\ldots\ldots+1}=1+\frac{z^v+z^{v+1}+\ldots\ldots+z^{u-1}}{z^{v-1}+z^{v-2}+\ldots\ldots+1}\ldots\ldots(1).$$

If $z>1$, the fraction in the final member is less than $\dfrac{(u-v)z^v}{vz^{v-1}}$.

$$\therefore \frac{z^u-1}{z^v-1}>1+\frac{(u-v)z}{v}>1+\frac{u-v}{v}, \text{ namely, } \frac{u}{v}\ldots\ldots(2).$$

$$\therefore z^u>1+(z^v-1)\frac{u}{v};\ \text{or, } (1+a)^{u/v}>1+a\frac{u}{v}.$$

If $z<1$, the final member of (1) is less than $1+(u-v)z/v$, and the character of the inequalities in (2) is reversed; but, as z^v-1 negative, the inequalities

are again reversed on multiplying by z^v-1, and the result is the same as before.

THEOREM II. *If x lies between* 0 *and* 1, *then* $(1+a)^x<1+ax$.

Let $x=\frac{v}{u}$, $v<u$, and put $(1+a)^{v/u}=1+y$. Then

$$1+a=(1+y)^{u/v}>1+\frac{u}{v}y. \quad \text{(Theorem I)}.$$

$\therefore y<a\frac{v}{u}$; and, therefore, $(1+a)^{v/u}<1+a\frac{v}{u}$.

THEOREM III. *If x is any positive quantity, then* $(1+a)^{-x}>1-ax$.

First. Let x be greater than 1, and equal to u/v. Put $(1+a)^{-x}=1+y$. Then,

$$(1+a)^{-x}=\left(\frac{1}{1+a}\right)^{u/v}=\left(1-\frac{a}{1+a}\right)^{u/v}>1-\frac{a}{1+a}\frac{u}{v} \quad \text{(Theorem I)}.$$

$$\therefore 1+y>1-\frac{a}{1+a}\frac{u}{v};\ y>-\frac{a}{1+a}\frac{u}{v}>-a\frac{u}{v};$$

$$\text{and } (1+a)^{-x}>1-a\frac{u}{v}, \text{ or } >1-ax.$$

(If $a>0$, then $\frac{a}{1+a}<a$, and $-\frac{a}{1+a}>a$. If $a<0$ and equal to $-b$, then, since $1+a$ is supposed >0, $-\frac{a}{1+a}=\frac{b}{1-b}>b$, or $>-a$).

Second. Let x be less than 1, and equal to v/u. Then

$$(1+a)=(1+y)^{-(u/v)}>1-y\frac{u}{v} \text{ (by preceding case).}$$

Whence, as before,

$$-y\frac{u}{v}<a,\ y>-a\frac{v}{u}, \text{ or } \triangleright -ax.$$

THEOREM IV. *For all values of x greater than* 1 *or less than* 0, $(1+a)^x<1+ax(1+a)^{x-1}$; *and for all values of x between* 0 *and* 1, $(1+a)^x>1+ax(1+a)^{x-1}$.

If $x>1$, or <0, we have (Theorems I and III),

$$\left(\frac{1}{1+a}\right)^x=\left(1-\frac{a}{1+a}\right)^x>1-ax\frac{1}{1+a}.$$

$\therefore 1>(1+a)^x-ax(1+a)^{x-1}$, $(1+a)^x<1+ax(1+a)^{x-1}$.

The second part of the theorem is similarly proved, by the aid of Theorem II.*

THEOREM V. *For positive values of x, the functions* $\left(1+\frac{1}{x}\right)^x$ *and* $\left(1-\frac{1}{x}\right)^x$ *are both increasing functions.*

* The demonstration of the theorem just proved the same in principle as that given in Chrystal's *Textbook of Algebra*, and is given here merely for the sake of completeness.

Let x_1 and x_2 be two values of x; assume $x_1 > x_2$, and put $\left(1+\frac{1}{x_1}\right)^{x_1} \div \left(1+\frac{1}{x_2}\right)^{x_2} = \lambda$. Then,

$$\left(1+\frac{1}{x_1}\right)^{x_1} = \lambda\left(1+\frac{1}{x_2}\right)^{x_2},$$

whence

$$\lambda^{1/x_2}\left(1+\frac{1}{x_2}\right) = \left(1+\frac{1}{x_1}\right)^{x_1/x_2} > 1+\frac{1}{x_2} \text{ (Theorem I)};$$

and, therefore,

$$\lambda > 1, \text{ or } \left(1+\frac{1}{x_1}\right)^{x_1} > \left(1+\frac{1}{x_2}\right)^{x_2}.$$

Similarly for $\left(1-\frac{1}{x}\right)^x$.

It follows that $\left(1-\frac{1}{x}\right)^{-x}$ increases as x increases.

THEOREM VI. *The functions* $\left(1+\frac{1}{x}\right)^x$ *and* $\left(1-\frac{1}{x}\right)^x$ *have each a limit as x increases indefinitely, and these limits are the reciprocal of each other.*

For we have,

$$\left(1+\frac{1}{x}\right)^x\left(1-\frac{1}{x}\right)^x = \left(1-\frac{1}{x^2}\right)^x < 1, \text{ but} > 1-\frac{1}{x} \text{ (Theorem I).}$$

This shows that neither function can increase indefinitely, since they are both increasing functions (Theorem IV).

From the inequalities

$$1 > \left[1+\frac{1}{x}\right]^x\left[1-\frac{1}{x}\right]^x > 1-\frac{1}{x},$$

follows

$$\lim\left[1+\frac{1}{x}\right]^x \lim\left[1-\frac{1}{x}\right]^x = 1.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

181. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Show that $\phi(1)\dfrac{x}{1+x^2}-\phi(3)\dfrac{x^3}{1+x^6}+\phi(5)\dfrac{x^3}{1+x^{10}}-\text{ad inf.}=\dfrac{x(1-x^2)}{(1+x^2)^2}, \phi(n)$

being the number of integers less than n and prime to it. [From Hall and Knight's *Higher Algebra*, page 358].

Solution* by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

$$\phi(1)\ \frac{y}{1-y^2}+\phi(3)\frac{y^3}{1-y^6}+\phi(5)\frac{y^5}{1-y^{10}}+\text{..........}$$

$$=\frac{y}{1-y^2}+\frac{2y^3}{1-y^6}+\frac{4y^5}{1-y^{10}}+\frac{6y^7}{1-y^{14}}+\frac{8y^9}{1-y^{18}}+\text{.....}$$

$$=y(1+y^2+y^4+y^6+y^8+\text{.....})+2y^3(1+y^6+\text{.....})+4y^5(1+y^{10}+\text{.....})$$

$$+6y^7(1+y^{14}+\text{.....})+8y^9(1+y^{18}+\text{.....})+\text{.....}=y+3y^3+5y^5+7y^7+9y^9+\text{.....}$$

$$=y(1+y^2)(1+2y^2+3y^4+4y^5+5y^8+\text{.....})=y(1+y^2)(1-y^2)^{-2}=\frac{y(1+y^2)}{(1-y^2)^2}.$$

Let $y=x\surd(-1)$.

$$\therefore \phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-\phi(7)\ \frac{x^7}{1+x^{14}}+\text{.....}=\frac{x(1-x^2)}{(1+x^2)^2}.$$

182. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Find the values of x_1, x_2, x_3, ..., x_n which satisfy the following system of simultaneous equations:

$$\frac{x_1}{a_i-b_1}+\frac{x_2}{a_i-b_2}+\text{.....}+\frac{x_n}{a_i-b_n}=1\quad (i=1,\ 2,\ \text{.....},\ n).$$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

The equation $\dfrac{x_1}{\theta-b_1}+\dfrac{x_2}{\theta-b_2}+\text{.....}+\dfrac{x_n}{\theta-b_n}=1-\dfrac{(\theta-a_1)(\theta-a_2)\text{.....}(\theta-a_n)}{(\theta-b_1)(\theta-b_2)\text{.....}(\theta-b_n)}$,

is an equation of the $(n-1)$th degree in θ and is satisfied by the n values a_1, a_2, ..., a_n. Multiply each side by $\theta-b_1$, and then place $\theta=b_1$.

*The general term should be considered. For $8y^9$ read $6y^9$. Editor D.

$$\therefore x_1=-\frac{(b_1-a_1)(b_1-a_2)\ldots\ldots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\ldots\ldots(b_1-b_n)},\ x_r=-\frac{(b_r-a_1)(b_r-a_2)\ldots\ldots(b_r-a_n)}{(b_r-b_1)(b_r-b_2)\ldots\ldots(b_r-b_n)}.$$

GEOMETRY.

203. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Show that two parabolae cau always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex are parallel to the opposite side.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

In order that $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ may be a parabola we must have

$$l^2a^2+m^2b^2+n^2c^2=2ablm+2bcmn+2acln.$$

For tangency at a vertex we must have $l\beta+m\alpha=0$, also $\alpha\sin B+\beta\sin A=0$.

$\therefore l/m=\sin A/\sin B=a/b$, or $l=ma/b$.

$\therefore m^2a^2/b^2+m^2b^2+n^2c^2=2a^2m^2+2bcmn+2a^2cmn/b$.

$\therefore m^2(a^2-b^2)^2+n^2b^2c^2=(2b^3c+2a^2bc)mn$.

$\therefore n=\dfrac{(a\pm b)^2}{bc}m$.

$\therefore ac\beta\gamma+bc\gamma\alpha+(a\pm b)^2\alpha\beta=0$ is the equation to the two parabolae.

If two of the sides be taken as axes, the Cartesian equation is $(x\pm y)^2+bx+ay=0$. In order that the tangents at each vertex may be parallel to the opposite sides we must have $l/m=b/a$, $m/n=c/b$.

$\therefore \beta\gamma/a+\gamma\alpha/b+\alpha\beta/c=0$ is the conic required, and this conic is an ellipse.

II. Solution by G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let the triangle be formed by the lines α, β, γ. The tangent at the vertex $\alpha\beta$ to the circumcircle is

$$a\beta+b\alpha=0.$$

The equation of any circumconic having this line for a tangent may be written

$$a\beta\gamma+b\gamma\alpha+\lambda\alpha\beta=0.$$

The coördinates of the center, *i. e.*, of the pole of the line at infinity, are found to be

$$\lambda ac-a^3+ab^2,\quad \lambda bc-b^3+a^2b,\quad \lambda a^2+\lambda b^2-\lambda^2c.$$

If this point lies on the line at infinity, the conic will be a parabola. This condition gives $\lambda=(a\pm b)^2/c$, and hence there are two parabolae satisfying this condition. The coördinates of the centers become

$$2ab(b\pm a),\quad 2ab(a\pm b),\quad \mp 2ab(a\pm b)^2/c.$$

From the first two coördinates we see that the axes of these parabolae are the bisectors of the angle at $\alpha\beta$.

Problem 204 was also solved by D. B. Northrup, Schenectady, N. Y.

205. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Having given any two systems of conjugate semi-diameters of an ellipsoid, the parallelopiped which has any three for continuous edges is equal to that which has the other three for continuous edges.

Remark by H. B. LEONARD, A.B., Chicago, Ill.

Solved in C. Smith's "An Elementary Treatise on Solid Geometry," page 76.

CALCULUS.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $(y^2+x^2)=a^2(x^2-y^2)$ round the axis of x.

II. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

This equation represents singly the two lines $y=\pm x\sqrt{(a^2-1)/(a^2+1)}$, and the surface generated is simply a right circular cone whose area, volume, bounded by the plane $x=c$, is simply $\frac{1}{3}\pi c^3(a^2-1)/(a^2+1)$, and where $a\not\leq 1$.

168. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The tangent of what Cartesian curve makes an x-intercept always m times as long as as the corresponding y-intercept?

II. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let the tangent at P (x, y) meet the axes in the points A and B. Let the perpendicular from P to OY meet it at M. Call θ the angle APM.

Then $PB=m.PA$, $OM=m.AM$; *i. e.*, $y=m.PM.\tan\theta=mx(dy/dx)$.

$\therefore y^m=cx$.

AVERAGE AND PROBABILITY.

145. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

In each quadrant of a given circle, a circle is described at random. A point is taken at random in each of these circles. What is the average area of the quadrilateral formed by joining with straight lines these four points?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let P, Q, R, S be the centers of the four circles. The average area of the required quadrilateral is the same as the average area of the quadrilateral formed by joining the centroids of the four circles, or their centers.

Let $OP=s$, $PC=t$, $OQ=u$, $QF=v$, $OR=w$, $RE=x$, $OS=y$, $DS=z$, $OG=a$, $\angle AOP=\theta$, $\angle AOQ=\phi$, $\angle BOR=\psi$, $\angle BOS=\rho$.

The limits of t, v, x, and z are 0 and $a(\sqrt{2}-1)=m$; of s, $t\sqrt{2}$ and $a-t$; of u, $v\sqrt{2}$ and $a-v$; of w, $x\sqrt{2}$ and $a-x$; of y, $z\sqrt{2}$ and $a-z$; of θ, $\sin^{-1}(t/s)=\theta''$ and $\frac{1}{2}\pi-\theta''=\theta'$; of ϕ, $\sin^{-1}(v/u)=\phi''$ and $\frac{1}{2}\pi-\phi''=\phi'$; of ψ, $\sin^{-1}(x/w)=\psi''$ and $\frac{1}{2}\pi-\psi''=\psi'$; of ρ, $\sin^{-1}(z/y)=\rho''$ and $\frac{1}{2}\pi-\rho''=\rho'$.

The area of $PQRS=\frac{1}{2}[su\sin(\theta+\phi)+uw\sin(\phi+\psi)+wy\sin(\psi+\rho)+sy\sin(\theta+\rho)]=A$.

$$\therefore \triangle=\frac{\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{2}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta''}^{\theta'}\int_{\phi''}^{\phi'}\int_{\psi''}^{\psi'}\int_{\rho''}^{\rho'}A\,I}{\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{1}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta''}^{\theta'}\int_{\phi''}^{\phi'}\int_{\psi''}^{\psi'}\int_{\rho''}^{\rho'}I}$$

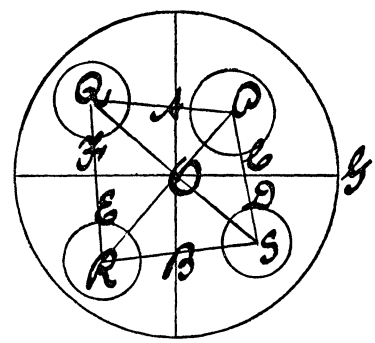
where $I=dt\,dv\,dx\,dz\,sds\,udu\,wdw\,ydy\,d\theta\,d\phi\,d\psi\,d\rho$.

$$\triangle=\frac{6^8}{(3\pi-28+16\sqrt{2})^4a^{12}}$$

$$\times\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{2}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta''}^{\theta'}\int_{\phi''}^{\phi'}\int_{\psi''}^{\psi'}\int_{\rho''}^{\rho'}A\,I$$

$$\triangle=\frac{648}{(3\pi-28+16\sqrt{2})^2a^6}\Big[\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{\theta''}^{\theta'}\int_0^m\int_{v\sqrt{2}}^{a-v}\int_{\phi''}^{\phi'}$$

$$\times s^2u^2\sin(\theta+\phi)dt\,ds\,d\theta\,dv\,du\,d\phi$$



$$+\int_0^m\int_{v\sqrt{2}}^{a-v}\int_{\phi''}^{\phi'}\int_0^m\int_{x\sqrt{2}}^{a-x}\int_{\psi''}^{\psi'}$$

$$\times u^2w^2\sin(\phi+\omega)dv\,du\,d\phi\,dx\,dw\,d\psi$$

$$+\int_0^m\int_{x\sqrt{2}}^{a-x}\int_{\psi''}^{\psi'}\int_0^m\int_{z\sqrt{2}}^{a-z}\int_{\rho''}^{\rho'}$$

$$\times w^2y^2\sin(\psi+\rho)dx\,dw\,d\psi\,dz\,dy\,d\rho$$

$$+\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{\theta''}^{\theta'}\int_0^m\int_{z\sqrt{2}}^{a-z}\int_{\rho''}^{\rho'}s^2y^2\sin(\theta+\rho)dt\,ds\,d\theta\,dz\,dy\,d\rho\Big]$$

$$=\frac{9(3729-2624\sqrt{2})a^2}{25(3\pi-28+16\sqrt{2})^2}=\left(\frac{3\ a}{5}\right)^2\left(\frac{32\sqrt{2}-41}{3\pi-28+16\sqrt{2}}\right)^2=\tfrac{2}{5}a^2, \text{ nearly,}$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

187. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Express by radicals the roots of $x^7+px^5+\frac{2}{7}p^2x^3+\frac{1}{49}p^3x+r=0$.

188. Proposed by GUY SCHUYLER.

$xy+ab=2ax,\quad x^2y^2+a^2b^2=2b^2y^2$.

GEOMETRY.

209. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find by a geometrical method the maximum value of $\sin\theta\,\cos\theta\,\cos2\theta$.

210. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Let ADC be a triangle with angle $C=120°$, and let the interior bisector of angle C meet AD in B. Prove that $2.CB$ is the harmonic mean between CA and CD.

211. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Prove the validity of the following construction of an inscribed regular pentagon and regular decagon: Draw any two perpendicular radii of the given circle with center C. Call E the end of one radius CE and M the middle point of the perpendicular radius CM. Take the point R on CM produced through C such that $RCM=EM$. Then RC=side of inscribed regular decagon, RE=side of inscribed regular pentagon.

CALCULUS.

172. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $x\dfrac{dy}{dx}=\dfrac{y}{y^{-1}-\log x}$.

DIOPHANTINE ANALYSIS.

119. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If p be any prime number and n any positive integer, the congruence

$x^{p^n} \equiv x (\mathrm{mod} p^n)$ has p and p solutions modp^n. Hence the congruence defines the Galois field of order p^n if and only if $n=1$.

AVERAGE AND PROBABILITY.

148. Proposed by M. C. RORTY, Boston, Mass.

Assuming n points to fall at random upon a circle of circumference a, what is the probability of m or more points falling within a length b upon this circumference.

NOTE. This problem has practical application in determining the probability of accidental rushes of telephone calls as distinct from those rushes which are due to commercial causes. The solution for m or more points falling within a specified length b is known. The problem presented above differs from this in that a solution is required for *any* length b.

149. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Three points are taken at random on the convex surface of a right cone. Find the probability that the section of the cone made by the plane passing through them is a complete ellipse.

MISCELLANEOUS.

142. Proposed by R. A. WELLS, Franklin College, New Athens, O.

Find a general expression for the value of θ such that when θ is one of the acute angles of a right triangle, the three sides of the triangle will be commensurable.

NOTES.

Dr. C. N. Haskins has been appointed Instructor in Mathematics at Yale University.

Dr. A. B. Pierce has been appointed Instructor in Mathematics at the University of Michigan.

Dr. G. B. Halsted has been elected Professor of Mathematics at Kenyon College, Gambier, O.

Mr. A. C. Minear has been appointed Professor of Mathematics in the University of Southern California, to succeed Mr. Paul Arnold, resigned.

Dr. H. C. DeMott, Principal of the Preparatory Department of the Illinois Wesleyan University, has been promoted to the University chair of mathematics.

In line 14 of Professor Himel's Note in the October number, read "Thus, in the above example, if 2 be true."

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ERRATA.

Page 78, for No. 106, read 112.
Page 115, next to last line, delete $-.0095M\sin^2\varphi$.
Page 117, last line, for Bulletin American Mathematical Society, read Science.
Page 243, ll. 15 and 24, and p. 244, l. 7, for $(-b\pm\sqrt{D})/a$, read $(-b\pm\sqrt{D})/2a$.
Page 243, lines 21, 25, 31, for $2(|a|+\sqrt{D})$, read $2|a|+\sqrt{D}$.
Page 244, strike out 2 where it appears as a coefficient of $\sqrt{D}$.
Page 252, first line, read has p and only p solutions.

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A GENERALIZATION OF SYMMETRIC AND SKEW-SYMMETRIC DETERMINANTS.

By DR. L. E. DICKSON, The University of Chicago.

1. Examples of symmetric and skew-symmetric determinants are, respectively,

$$Y=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix},\quad K=\begin{vmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{vmatrix}.$$

A direct generalization is afforded by the following determinant:

$$\triangle=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} \end{vmatrix}.$$

For determinants like $\triangle$, and D below, with a_{ij} and A_{ij} real, we have the

THEOREM. *A determinant with symmetrically conjugate elements is real.*

For proof we reflect $\triangle$ on the main diagonal and obtain an equal determinant whose elements are conjugate with the corresponding ones of the given determinant. From $\triangle=\bar{\triangle}$ follows $\triangle=$real.

2. Expanding $\triangle$, we get $\triangle=Y+Q$, where

$$Q\equiv -a_{11}A^2_{23}-a_{22}A^2_{13}-a_{33}A^2_{12}+2a_{12}A_{13}A_{23}+2a_{23}A_{12}A_{13}-2a_{13}A_{12}A_{23}.$$

Regarding A_{12}, A_{13}, and A_{23} as variables, we seek the conditions under which Q shall factor into linear expressions. To fix the ideas, let $a_{11}\neq 0$. Completing the square in A_{23}, we get finally

$$-a_{11}Q=(a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12})^2-A^2_{13}(a^2_{12}-a_{11}a_{22})$$
$$-A^2_{12}(a^2_{13}-a_{11}a_{33})-2A_{12}A_{13}(a_{11}a_{23}-a_{12}a_{13}).$$

The last three terms, with their signs changed, form a perfect square if and only if

$$(a^2_{12}-a_{11}a_{22})(a^2_{13}-a_{11}a_{33})-(a_{11}a_{23}-a_{12}a_{13})^2=0,$$

that is, if $a_{11}Y=0$. For $Y=0$ we therefore have

$$-a_{11}Q=(P+T)(P-T),\quad P=a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12},$$
$$T=A_{13}\sqrt{(a^2_{12}-a_{11}a_{22})}+A_{12}\sqrt{(a^2_{13}-a_{11}a_{33})},$$

the signs of the radicals being suitably chosen.

Interpreting the A_{ij} as homogeneous point-coördinates in the plane, we conclude that the equation $\triangle=Y$ represents a pair of straight lines if and only if $Y=0$.

3. Consider next the symmetrically conjugate determinant

$$D=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} & a_{14}+iA_{14} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} & a_{24}+iA_{24} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} & a_{24}+iA_{34} \\ a_{14}-iA_{14} & a_{24}-iA_{24} & a_{34}-iA_{34} & a_{44} \end{vmatrix},$$

and the included symmetric and skew-symmetric determinants

$$a=|\,a_{ij}\,|,\quad A=\begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{vmatrix}.$$

By the theory of skew-symmetric determinants,

$$K\equiv 0,\quad A\equiv(A_{12}A_{34}-A_{13}A_{24}+A_{14}A_{23})^2.$$

To expand D, we write $a_{ii}=a_{ii}+0$, thus obtaining each element expressed as an algebraic sum of two terms. Hence D equals the sum of 16 determinants of the fourth order, among which occur a and A. In view ot the theorem of §1, 8 of the determinants cancel, namely, those in which the elements of an *odd* number of columns contain the factor i. Each of the remaining 6 determinants have the factor i^2. Their sum is

$$-\begin{vmatrix} a_{11} & a_{12} & A_{13} & A_{14} \\ a_{12} & a_{22} & A_{23} & A_{24} \\ a_{13} & a_{23} & 0 & A_{34} \\ a_{14} & a_{24} & -A_{34} & 0 \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & A_{13} & a_{14} \\ a_{12} & 0 & A_{23} & a_{24} \\ a_{13} & -A_{23} & 0 & a_{34} \\ a_{14} & -A_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & a_{13} & A_{14} \\ a_{12} & 0 & a_{23} & A_{24} \\ a_{13} & -A_{23} & a_{33} & A_{34} \\ a_{14} & -A_{24} & a_{34} & 0 \end{vmatrix}$$

$$-\begin{vmatrix} 0 & a_{12} & a_{13} & A_{14} \\ -A_{12} & a_{22} & a_{23} & A_{24} \\ -A_{13} & a_{23} & a_{33} & A_{34} \\ -A_{14} & a_{24} & a_{34} & 0 \end{vmatrix} - \begin{vmatrix} 0 & a_{12} & A_{13} & a_{14} \\ -A_{12} & a_{22} & A_{23} & a_{24} \\ -A_{13} & a_{23} & 0 & a_{34} \\ -A_{14} & a_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} 0 & A_{12} & a_{13} & a_{14} \\ -A_{12} & 0 & a_{23} & a_{24} \\ -A_{13} & -A_{23} & a_{33} & a_{34} \\ -A_{14} & -A_{24} & a_{34} & a_{44} \end{vmatrix}$$

To the first of the six we apply Laplace's expansion, the dividing line separating the first two *columns* from the last two. For the other five, we first interchange columns to bring the a_{ij} into the first and second columns and then apply Laplace's expansion. In their final sum, the coefficient of $-A_{12}A_{34}$ is

$$\begin{vmatrix} a_{12} & a_{14} \\ a_{23} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{13} \\ a_{24} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{23} \\ a_{14} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{24} \\ a_{13} & a_{34} \end{vmatrix} = 2\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

Similar simplifications may be made in the coefficients of $A_{13}A_{24}$ and $A_{14}A_{23}$. A term $A_{ij}{}^2$ occurs singly. A term $A_{ij}A_{kl}$ in which two and but two of the subscripts i, j, k, l are equal occurs twice, with equal coefficients. We obtain the final result $D=a+A-S$, where (denoting a_{ij} by ij)

$$S = A^2_{12}\begin{vmatrix} 33 & 34 \\ 34 & 44 \end{vmatrix} + A^2_{13}\begin{vmatrix} 22 & 24 \\ 24 & 44 \end{vmatrix} + A^2_{14}\begin{vmatrix} 22 & 23 \\ 23 & 33 \end{vmatrix} + A^2_{23}\begin{vmatrix} 11 & 14 \\ 14 & 44 \end{vmatrix}$$

$$+A^2_{24}\begin{vmatrix} 11 & 13 \\ 13 & 33 \end{vmatrix} + A^2_{34}\begin{vmatrix} 11 & 12 \\ 12 & 22 \end{vmatrix} + 2A_{12}A_{34}\begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} + 2A_{13}A_{24}\begin{vmatrix} 12 & 14 \\ 23 & 34 \end{vmatrix}$$

$$+2A_{14}A_{23}\begin{vmatrix} 12 & 13 \\ 24 & 34 \end{vmatrix} - 2A_{12}A_{13}\begin{vmatrix} 23 & 24 \\ 34 & 44 \end{vmatrix} + 2A_{12}A_{14}\begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix}$$

$$-2A_{12}A_{24}\begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} - 2A_{13}A_{14}\begin{vmatrix} 22 & 24 \\ 23 & 34 \end{vmatrix} - 2A_{13}A_{23}\begin{vmatrix} 12 & 14 \\ 24 & 44 \end{vmatrix}$$

$$-2A_{13}A_{34}\begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} - 2A_{14}A_{24}\begin{vmatrix} 12 & 13 \\ 23 & 33 \end{vmatrix} + 2A_{14}A_{34}\begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix}$$

$$-2A_{23}A_{24}\begin{vmatrix} 11 & 13 \\ 14 & 34 \end{vmatrix} - 2A_{24}A_{34}\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix} + 2A_{12}A_{23}\begin{vmatrix} 13 & 14 \\ 34 & 44 \end{vmatrix}$$

$$+2A_{23}A_{34}\begin{vmatrix} 11 & 14 \\ 12 & 24 \end{vmatrix}.$$

A law for the simplified expansion of D, which may be readily extended to like determinants of any order, may now be given. The law may be expressed very luminously by use of a "multiplication table." As in the usual notation

for quadratic forms, a term $2cxy$ is conveniently expressed by $cxy+cyx$. Then S is given by the accompanying multiplication table.

	A_{34}	A_{24}	A_{23}	A_{14}	A_{13}	A_{12}
A_{34}	$\begin{vmatrix}11&12\\12&22\end{vmatrix}$	$-\begin{vmatrix}11&13\\12&23\end{vmatrix}$	$\begin{vmatrix}11&14\\12&24\end{vmatrix}$	$\begin{vmatrix}12&13\\22&23\end{vmatrix}$	$-\begin{vmatrix}12&14\\22&24\end{vmatrix}$	$\begin{vmatrix}13&14\\23&24\end{vmatrix}$
A_{24}	$-\begin{vmatrix}11&13\\12&23\end{vmatrix}$	$\begin{vmatrix}11&13\\13&33\end{vmatrix}$	$-\begin{vmatrix}11&14\\13&34\end{vmatrix}$	$-\begin{vmatrix}12&13\\23&33\end{vmatrix}$	$\begin{vmatrix}12&14\\23&34\end{vmatrix}$	$-\begin{vmatrix}13&14\\33&34\end{vmatrix}$
A_{23}	$\begin{vmatrix}11&12\\14&24\end{vmatrix}$	$-\begin{vmatrix}11&13\\14&34\end{vmatrix}$	$\begin{vmatrix}11&14\\14&44\end{vmatrix}$	$\begin{vmatrix}12&13\\24&34\end{vmatrix}$	$-\begin{vmatrix}12&14\\24&44\end{vmatrix}$	$\begin{vmatrix}13&14\\34&44\end{vmatrix}$
A_{14}	$-\begin{vmatrix}12&22\\13&23\end{vmatrix}$	$-\begin{vmatrix}12&23\\13&33\end{vmatrix}$	$\begin{vmatrix}12&24\\13&34\end{vmatrix}$	$\begin{vmatrix}22&23\\23&33\end{vmatrix}$	$-\begin{vmatrix}22&24\\23&34\end{vmatrix}$	$\begin{vmatrix}23&24\\33&34\end{vmatrix}$
A_{13}	$-\begin{vmatrix}12&22\\14&24\end{vmatrix}$	$\begin{vmatrix}12&23\\14&34\end{vmatrix}$	$-\begin{vmatrix}12&24\\14&44\end{vmatrix}$	$-\begin{vmatrix}22&23\\24&34\end{vmatrix}$	$\begin{vmatrix}22&24\\34&44\end{vmatrix}$	$-\begin{vmatrix}23&24\\34&44\end{vmatrix}$
A_{12}	$\begin{vmatrix}13&23\\14&24\end{vmatrix}$	$-\begin{vmatrix}13&33\\14&34\end{vmatrix}$	$\begin{vmatrix}13&34\\14&44\end{vmatrix}$	$\begin{vmatrix}23&33\\24&34\end{vmatrix}$	$-\begin{vmatrix}23&34\\24&44\end{vmatrix}$	$\begin{vmatrix}33&34\\34&44\end{vmatrix}$

The law of formation is now evident. The body of the table gives in proper position and with proper sign the minors of $|a_{ij}| \equiv a$. It follows (Muir, *Theory of Determinants*, §174) that the determinant of the sixth order defined by the table equals a^3, being the second compound of a.

4. Changing the notation by replacing A_{13} and A_{24} by their negatives, we obtain from S a quadratic form S' with the signs of its terms all positive, while A remains unaltered. The body of the multiplication table is altered only in having all its signs made positive. If $a \neq 0$, the resulting matrix M defines a linear transformation on the six variables A_{ij}. It is known (author's *Linear Groups*, pp. 145-155) to leave invariant the quadratic function

$$F \equiv A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23},$$

whose square equals A (§3). Of several geometrical interpretations, the most elegant consists in regarding the A_{ij} as the six line-coödinates in ordinary space, so that $F \equiv 0$. The matrix M therefore defines a transformation of the straight lines in space. We may also interpret the equation $D=a$ (or $S=0$) as the totality of straight lines whose six line-coördinates are subject to a quadratic condition.

NOTE ON THE NECESSARY CONDITION THAT TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS SHALL HAVE COMMON INTEGRALS.*

By IDA MAY SCHOTTENFELS.

Professor Von Escherich in the *Denkschriften der Wiener Akademie*, Vol. 46, and later Heffter in Crêlle's *Journal*, Vol. 116, proved that there exists for linear differential homogeneous equations a theory analogous to that of algebraic equations, confining their researches to the analogues to theorems upon the Highest Common Factor and Lowest Common Multiple.

During the past year Dr. Epsteen and Dr. Pierce revived this subject, and in THE AMERICAN MATHEMATICAL MONTHLY, Vol. X, March, 1903, pp. 63-68, Dr. Epsteen gives the necessary condition for one common integral, while Dr. Pierce gives the sufficient condition for $k \gtreqless 1$ independent common integrals.

This note gives the necessary condition for two common integrals.

The following example illustrates the method.

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \tag{1}$$

$$b_0x^3+b_1x^2+b_2x+b_3=0. \tag{2}$$

To find the necessary condition that equations (1) and (2) have two roots in common. According to the well known dialytic method of elimination of Sylvester we may write

$$\left\{\begin{array}{l} a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x \qquad =0, \\ \qquad a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \\ b_0x^5+b_1x^4+b_2x^3+b_3x^2 \qquad\qquad =0, \\ \qquad b_0x^4+b_1x^3+b_2x^2+b_3x \qquad =0, \\ \qquad\qquad b_0x^3+b_1x^2+b_2x+b_3=0. \end{array}\right. \tag{3}$$

Eliminating x^5, x^4, x^3, x^2, from (3) we get

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4x+0 \\ 0 & a_0 & a_1 & a_2 & a_3x+a_4 \\ b_0 & b_1 & b_2 & b_3 & 0x+0 \\ 0 & b_0 & b_1 & b_2 & b_3x+0 \\ 0 & 0 & 0 & b_1 & b_2x+b_3 \end{vmatrix} =0. \tag{4}$$

Since (4) shall hold for at least two values of x, and yet is linear in x, it is an identity in x, and hence

* Presented to the American Mathematical Society (New York) October 30, 1903.

$$
(5)\qquad
\begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 \\
0 & a_0 & a_1 & a_2 & a_3 \\
b_0 & b_1 & b_2 & b_3 & 0 \\
0 & b_0 & b_1 & b_2 & b_3 \\
0 & 0 & 0 & b_1 & b_2
\end{vmatrix} \equiv 0, \qquad
\begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & 0 \\
0 & a_0 & a_1 & a_2 & a_4 \\
b_0 & b_1 & b_2 & b_3 & 0 \\
0 & b_0 & b_1 & b_2 & 0 \\
0 & 0 & 0 & b_1 & b_3
\end{vmatrix} \equiv 0.
$$

Hence the required necessary condition that equations (1) and (2) have two roots in common is the vanishing of determinants (5).

The generalizatisn to equations of degree m aud n, that have p roots in common follows readily. The equation of degree m must be multiplied by x^{n-p} while the one of degree n must be multiplied by x^{m-p}.

In exactly the same manner the analog of this method for linear homogeneous differential equations can be developed. The theorem that a linear homogeneous differential equation of the nth order has n and only n linearly independent integrals is employed, and the necessary condition is expressed in linear homogeneous differential operators, and has a form analogous to that of the necessary condition (5).

Given the following two linear homogeneous differential equations:

$$
\text{(I)}\qquad \alpha_0(x)y^{\text{iv}}+\alpha_1(x)y'''+\alpha_2(x)y''+\alpha_3(x)y'+\alpha_4(x)y=0,
$$

$$
\text{(II)}\qquad \beta_0(x)y'''+\beta_1(x)y''+\beta_2(x)y'+\beta_3(x)y=0.
$$

If these two equations I and II have at least one common integral, we can by differentiation write,

$$
\text{(III)}\left\{
\begin{aligned}
&\alpha_0 y^{\text{v}}+(\alpha_0'+\alpha_1)y^{\text{iv}}+(\alpha_1'+\alpha_2)y'''+(\alpha_2'+\alpha_3)y''+(\alpha_3'+\alpha_4)y'+\alpha_4'\cdot y\equiv 0\\
&\qquad \alpha_0 y^{\text{iv}}+\alpha_1 y'''+\alpha_2 y''+\alpha_3 y'+\alpha_4\cdot y\equiv 0\\
&\beta_0 y^{\text{v}}+(2\beta_0'+\beta_1)y^{\text{iv}}+(\beta_0''+2\beta_1'+\beta_2)y'''+(\beta_1''+2\beta_2'+\beta_3)y''\\
&\qquad\qquad +(\beta_2''+2\beta_3')y'+\beta_3''\cdot y\equiv 0\\
&\qquad \beta_0 y^{\text{iv}}+(\beta_0'+\beta_1)y'''+(\beta_1'+\beta_2)y''+(\beta_2'+\beta_3)y'+\beta_3'\cdot y\equiv 0\\
&\qquad\qquad \beta_0 y'''+\beta_1 y''+\beta_2 y'+\beta_3\cdot y\equiv 0.
\end{aligned}
\right.
$$

Eliminating y^{v}, y^{iv}, y''', y'' from (III) we get

$$
\text{(IV)}\qquad
\begin{vmatrix}
\alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & (\alpha_3'+\alpha_4)y'+\alpha_4'\cdot y \\
0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 y'+\alpha_4\cdot y \\
\beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & (\beta_2''+2\beta_3')y'+\beta_3''\cdot y \\
0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & (\beta_2'+\beta_3)y'+\beta_3'\cdot y \\
0 & 0 & \beta_0 & \beta_2 & \beta_2 y'+\beta_3 y
\end{vmatrix} \equiv 0
$$

which may be written $\lambda y'+\mu y=0$.

If there are two independent common integrals, IV must hold for two linearly independent functions y (one not a constant times the other), whence $\lambda=0$, $\mu=0$, or

$$(\mathrm{V}_1)\quad \begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & \alpha_3'+\alpha_4 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_2''+2\beta_3' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} \equiv 0,$$

$$(\mathrm{V}_2)\quad \begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & \alpha_4' \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_4 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_3'' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_3' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_3 \end{vmatrix} \equiv 0.$$

Hence V_1 and V_2 furnish the required necessary condition.

NEW YORK CITY, *September, 1903.*

LINEAR COVARIANTS OF THE BINARY QUADRATIC AND CUBIC.

By L. C. WALKER, Professor of Mathematics, Colorado School of Mines, Golden, Col.

The definition of *weight* is that every coefficient is of weight w measured by its suffix, and that every product of coefficients is of weight measured by the sum of the suffixes of its various factors.

A semi-covariant of the two quantics is a function of the two sets of coefficients, which is homogeneous in each set separately, and *isobaric* (equal weight) on the whole, though not necessarily in the sets separately.

The practice of speaking of a covariant whose dimensions are partial degrees i_1, i_2 in the two sets of coefficients and ω in the variables has of late become almost universal.*

The degrees of quantics in the variables are generally† spoken of as their orders p_1, p_2.

The order ω, the partial degrees i_1, i_2 in the coefficients of the binary quadratic and cubic

$$(a_0,\, a_1,\, a_2)(x,\, y)^2,\quad (b_0,\, b_1,\, b_2,\, b_3)(x,\, y)^3,$$

and the weight w of the semi-invariant which is the leading coefficient C_0 in the linear covariant, are connected by the relation $i_1p_1+i_2p_2-\omega=2w$.

Here $p_1=2$, $p_2=3$, $\omega=1$, whence $2i_1+3i_2-1=2w$. More generally, if m be any positive integer and n any positive odd integer, we have, from the conditions of linear covariancy, $2mi_1+3ni_2-1=2w$. Thus the binary quadratic and cubic have an indefinite number of linear covariants.

---

*Elliott's *Algebra of Quantics*. †*Ibid.*

We now shall find the linear covariants,

$$(a)\quad (2;\,1,\,2;\,1,\,3); \qquad (b)\quad (3;\,2,\,2;\,1,\,3);$$
$$(c)\quad (5;\,1,\,2;\,3,\,3); \qquad (d)\quad (6;\,2,\,2;\,3,\,3).$$

(a). Assume for the semi-invariant the most general form

$$S \equiv C_0 \equiv a_2 b_2 + \lambda_1 a_1 b_1 + \lambda_2 a_2 b_0,$$

where λ_1, λ_2 are arbitrary multipliers. Operate on this with the two annihilators

$$\Omega_1 \equiv a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2}, \quad \Omega_2 \equiv b_0 \frac{d}{db_1} + 2b_1 \frac{d}{db_2} + 3b_2 \frac{d}{db_3},$$

and we obtain

$$a_0 b_1 [\lambda_1 + 2] + a_1 b_0 [2\lambda_2 + \lambda_1];$$

for which to vanish, we must have

$$\lambda_1 + 2 = 0, \quad 2\lambda_2 + \lambda_1 = 0; \quad i.\,e.,\ \lambda_1 = -2\lambda_2 = -2.$$

$$\therefore C_0 \equiv a_0 b_2 - 2a_1 b_1 + a_2 b_0,$$
$$C_1 \equiv O_1 C_0 + O_2 C_0 \equiv a_0 b_3 - 2a_1 b_2 + a_2 b_1,$$

where we have used the annihilators

$$O_1 \equiv 2a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1}, \qquad O_2 \equiv 3b_1 \frac{d}{db_0} + 2b_2 \frac{d}{db_1} + b_3 \frac{d}{db_2}.$$

Thus the linear covariant is

I. $$(a_0 b_2 - 2a_1 b_1 + a_2 b_0)x + (a_0 b_3 - 2a_1 b_2 + a_2 b_1)y.$$

The second transvectant of the quadratic and cubic gives I.

(b). Including all possible terms, the semi-invariant is of the form

$$S \equiv C_0 \equiv a_0^2 b_3 + \lambda_1 a_0 a_1 b_2 + \lambda_2 a_0 a_2 b_1 + \lambda_3 a_1^2 b_1 + \lambda_4 a_1 b_2 b_0.$$

Operate on this with Ω_1, Ω_2. The vanishing of the expression requires four linear equations in the λ's, from which we find

$$\lambda_1 = -3, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = -1.$$

$$\therefore S \equiv C_0 \equiv a_0^2 b_3 - 3a_0 a_1 b_2 + a_0 a_2 b_1 + 2a_1^2 b_1 - a_1 a_2 b_0,$$
$$C_1 \equiv O_1 C_0 + O_2 C_0 \equiv -(a_2^2 b_0 - 3a_1 a_2 b_1 + a_0 a_2 b_2 + 2a_1^2 b_2 - a_0 a_1 b_3).$$

The linear covariant is

II. $$C_0x+C_1y.$$

The first transvectant of I and the quadratic gives II. The third transvectant of the cubic and the square of the quadratic gives II.

(c). Assume the semi-invariant to be

$$S\equiv C_0\equiv a_0b_0b_2b_3+\lambda_1a_0b_1^2b_3+\lambda_2a_0b_1b_2^2+\lambda_3a_1b_0b_1b_3$$
$$+\lambda_4a_1b_0b_2^2+\lambda_5a_1b_1^2b_2+\lambda_6a_2b_0^2b_3+\lambda_7a_2b_0b_1b_2+\lambda_8a_2b_1^2.$$

Solving as in (a), we obtain eight linear equations in the λ's, from which we find

$$\lambda_1=-2,\ \lambda_2=1,\ \lambda_3=2,\ \lambda_4=-4,\ \lambda_5=2,\ \lambda_6=-1,\ \lambda_7=3,\ \lambda_8=-2.$$

Thus the linear covariant is

III. $$(a_0b_0b_2b_3-2a_0b_1^2b_3+a_0b_1b_2^2+2a_1b_0b_1b_3-4a_1b_0b_2^2$$
$$+2a_1b_1^2b_2-a_2b_0^2b_3+3a_2b_0b_1b_2-2a_2b_1^3)x$$
$$-(a_2b_0b_1b_3-2a_2b_0b_2^2+a_2b_1^2b_2+2a_1b_0b_2b_3-4a_1b_1^2b_3$$
$$+2a_1b_1b_2^2-a_0b_0b_3^2+3a_0b_1b_2b_3-2a_0b_2^3)y.$$

The first transvectant of I and the *Hessian* of the cubic gives III. The second transvectant of the quadratic and the *cubicovariant* of the cubic gives III.

(d). Here we assume the semi-invariant to be

$$S\equiv C_0\equiv a_0^2b_0b_3^2+\lambda_1a_0^2b_2^3+\lambda_2a_0^2b_1b_2b_3+\lambda_3a_0a_1b_0b_2b_3+\lambda_4a_0a_1b_1^2b_3$$
$$+\lambda_5a_0a_1b_1b_2^2+\lambda_6a_0a_2b_0b_1b_3+\lambda_7a_0a_2b_0b_2^2+\lambda_8a_0a_2b_1^2b_2$$
$$+\lambda_9a_1^2b_0b_1b_3+\lambda_{10}a_1^2b_0b_2^2+\lambda_{11}a_1^2b_1^2b_2+\lambda_{12}a_1a_2b_0^2b_3$$
$$+\lambda_{13}a_1a_2b_0b_1b_2+\lambda_{14}a_1a_2b_1^3+\lambda_{15}a_2^2b_0b_2+\lambda_{16}a_2^2b_0b_1^2,$$

which includes all possible terms. As in (a), we obtain

$$a_0^2b_1^2b_3(\lambda_4+2\lambda_1)+a_0a_2b_0^2b_3(\lambda_{12}+\lambda_6)+a_0a_2b_1^3(\lambda_{14}+2\lambda_8)+a_1^2b_0^2b_3(2\lambda_{12}+\lambda_9)$$
$$+a_1^2b_1^3(2\lambda_{14}+2\lambda_{11})+a_2^2b_0^2b_1(4\lambda_{16}+4\lambda_{15})+a_0^2b_0b_2b_3(\lambda_3+\lambda_1+6)+a_0^2b_1b_2^2$$
$$(\lambda_5+6\lambda_2+3\lambda_1)+a_1a_2b_0^2b_2(4\lambda_{15}+\lambda_{12}+3\lambda_{12})+a_1a_2b_0b_1^2(4\lambda_{16}+3\lambda_{14}+2\lambda_{13})+$$
$$a_0a_1b_0b_1b_3(2\lambda_9+2\lambda_6+2\lambda_4+2\lambda_3)+a_0a_1b_0b_2^2(2\lambda_{10}+2\lambda_7+\lambda_5+3\lambda_3)+a_0a_1b_1^2b_2$$
$$(2\lambda_{11}+2\lambda_8+4\lambda_5+3\lambda_4)+a_0a_2b_0b_1b_2(\lambda_{13}+2\lambda_8+4\lambda_7+3\lambda_6)+a_1^2b_0b_1b_2(2\lambda_{13}+2\lambda_{11}$$
$$+4\lambda_{10}+3\lambda_9).$$

Its vanishing gives (5; 2, 2; 3, 3) relations which have to be satisfied by the (6; 2, 2; 3, 3) multipliers. If then (6; 2, 2; 3, 3) exceeds (5; 2, 2; 3, 3) we

can satisfy them, the number of the multipliers still arbitrary being $(6;\ 2,\ 2;\ 3,\ 3)-(5;\ 2,\ 2;\ 3,\ 3)\equiv 17-15=2$.

First, the first transvectant (1), of the cubic and the square of I; and (2), of the *Hessian* of the cubic and II: or, *second*, the second transvectant (1), of the cubic and the square of I; (2), of the quadratic and the product of I by the *Hessian* of the cubic; and (3), of the *Hessian* of the cubic and the product of the quadratic by I: or, *third*, the third transvectant of the *cubicovariant* of the cubic and the square of the quadratic—either *first* or *second* or *third* shows that the partitions* $a_2^2b_0b_2$ and $a_2^2b_0b_1^2$ are absent from this linear covariant. We then have $\lambda_{16}=\lambda_{15}=0$. Now from the other fourteen linear equations in the λ's we obtain

$$\lambda_1=2,\quad \lambda_2=\lambda_3=-3,\quad \lambda_4=6,\quad \lambda_5=-3,\quad \lambda_6=-1,\quad \lambda_7=2,$$
$$\lambda_8=-1,\quad \lambda_9=-2,\quad \lambda_{10}=4,\quad \lambda_{11}=-2,\quad \lambda_{12}=1,\quad \lambda_{13}=-3,\quad \lambda_{14}=2.$$

The required linear covariant is

$$\begin{aligned}&(a_0^2b_0b_3-3a_0^2b_1b_2b_3+2a_0^2b_2^3-3a_0a_1b_0b_2b_3+6a_0a_1b_1^2b_3-3a_0a_1b_1b_2^2\\&-a_0a_2b_0b_1b_3+a_0a_2b_0b_2^2-a_0a_2b_1^2b_2-2a_1^2b_0b_1b_3+4a_1^2b_0b_2^2-2a_1^2b_1^2b_2\\&+a_1a_2b_0^2b_3-3a_1a_2b_0b_1b_2+2a_1a_2b_1^3)x+(a_0a_1b_0b_3^2-3a_0a_1b_1b_2b_3+2a_0a_1b_2^3\\&-a_0a_2b_0b_2b_3+2a_0a_2b_1^2b_2-a_0a_2b_1b_2^2-2a_1^2b_0b_2b_3+4a_1^2b_1^2b_3-2a_1^2b_1b_2^2\\&-3a_1a_2b_0b_1b_3+6a_1a_2b_0b_1^2-3a_1a_2b_1^2b_2+a_2^2b_0^2b_3-3a_2^2b_0b_1b_2+2a_2^2b_1^3)y.\end{aligned}$$

The number of linearly independent semi-invariants of given weight w and partial degrees i_1, i_2 of the quadratic and cubic, is given by

$$(w;\ i_1,\ p_1;\ i_2,\ p_2)-(w-1;\ i_1;\ p_1;\ i_2,\ p_2).$$

This expression is not applicable when it exceeds unity for all values of m and n that do not give linear semi-invariants, because to each of these values corresponds one and only one linear covariant for a transvection of some combination (1), of the two quantics; or (2), of their covariants; or (3), of the quantics and their covariants. Professor Paul Gordan has proved that a complete system of transvectants is coextensive with a complete system of covariants, also including invariants as a particular case. For the geometrical interpretation of this system of quantics, see Art. 198 of Salmon's *Higher Algebra*.

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* For the theory of numbers of partitions, see Professor Cayley's second memoir on Quantics (*Collected Works*, Vol. II).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

169. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

At what rate per cent. must a note be discounted at the end of a quarter of a year in order to produce a discount equivalent to 10% interest for the year?

No solution received.

170. Proposed by J. F. LAWRENCE, A. B., Breckenridge, Mo.

Suppose the market value of 5% bank stock to be 11 1-9% higher than 8% corporation bonds; I realize 8% on my investment, and my income from each is \$180. What did I invest in each?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

$11\frac{1}{9}\%=\frac{1}{9}$. $\therefore$ The 5% stock costs $\frac{10}{9}$ as much per share as the 8% stock.

$$\therefore \frac{\text{Investment in } 5\% \text{ stock}}{\frac{10}{9}}\times\tfrac{5}{100}=\text{Investment in } 8\% \text{ stock}\div\tfrac{8}{100}.$$

$\therefore$ Investment in 5% stock : Investment in 8% stock $=16:9$.

Total investment $=\$360\div.08=\$4500$.

Investment in 5% stock $=\frac{16}{25}$ of \$4500 $=\$2880$.

Investment in 8% stock $=\frac{9}{25}$ of \$4500 $=\$1620$.

Also solved by *H. B. LEONARD, B. S.*, and *G. W. GREENWOOD, B. A.*

171. Proposed by JOHN S. ROYER, Editor of The School Visitor, Columbus, Ohio.

A drawer made of inch boards is 8 inches wide, 6 inches deep, and slides horizontally. How far must it be drawn out to put into it a book 4 inches thick, 6 inches wide, and 9 inches long?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and H. B. LEONARD, B. S.

Let a=length of book, b=its thickness, c=depth of drawer, x=distance it must be drawn out. If we draw out the drawer just far enough to put in the book lengthwise we have from the figure properly drawn, $a:x=c:b$ or $9:x=5:4$. Therefore $x=7\frac{1}{5}$ inches.

ALGEBRA.

183. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the condition that $x:y:z$ may be real, given that $\Sigma ax^2=\Sigma a\alpha x=0$ and $\Sigma a\alpha^2=1$.

Solution by H. B. LEONARD, B. S.

From $ax^2+by^2+cz^2=0=a\alpha x+b\beta y+c\gamma z$, $a\alpha^2+\beta^2 b+c\lambda^2=1$, we get

$$x^2(a\alpha^2+c\lambda^2)a+2a\alpha b\beta xy+y^2(b\beta^2+c\gamma^2)b=0,$$

$$\frac{x}{y}=-\frac{a\alpha b\beta\pm\gamma\sqrt{(-abc)}}{a(1-b\beta^2)},\quad \frac{x}{z}=-\frac{a\alpha c\lambda\pm\beta\sqrt{(-abc)}}{a(1-c\gamma^2)},$$

$$\frac{y}{z}=-\frac{b\beta c\gamma\pm\alpha\sqrt{(-abc)}}{b(1-a\alpha^2)}.$$

Assuming a, b, c, α, β, γ to be real, then in order that $x:y:z$ may be real, $\sqrt{(-abc)}$ must be real. From $ax^2+by^2+cz^2=0$, it is clear that a, b, c can not all have the same sign and hence we must have one of the quantities a, b, c negative and the other two positive.

184. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

If m rows, viz., the h_1th, h_2th, ..., h_mth, be transferred so as to become the 1st, 2nd, ..., mth, without altering the relative positions of the remaining rows, and that n columns, viz., the k_1th, k_2th, ..., k_nth, be similarly transformed the determinant thus obtained is the same as the original or differs from it only in sign according as $h_1+h_2+.....+h_m-\frac{1}{2}m(m+1)+k_1+k_2+.....+k_n-\frac{1}{2}n(n+1)$ is odd or even. [*Muir.*]

Solution by G. W. GREENWOOD, B. A. (Oxon), G. B. M. ZERR, A. M., Ph. D., and H. B. LEONARD, B. S.

In transferring the pth row (or column) to the qth row (or column) there are $p-q$ interchanges of adjacent rows (or columns) and therefore $p-q$ changes of sign. Hence, in the given example, there are

$$(h_1-1)+(h_2-2)+.....+(h_m-m)+(k_1-1)+(k_2-2)+.....+(k_n-n),$$
$$i.\ e.,\ h_1+h_2+.....+h_m-\tfrac{1}{2}m(m+1)+k_1+k_2+.....+k_n-\tfrac{1}{2}n(n+1)$$

changes of sign, and the determinant is unaltered in value, or differs only in sign, according as this value is *even* or *odd*; not odd or even as stated.

185. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Without introducing radicals, eliminate x and y from the equations (1) $ax^2+bx+c=0$, (2) $ay^2+by+d=0$, and (3) $ax^2y^2+bxy+e=0$.

I. Solution by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill., G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. E. SAUNDERS, Hackney, Ohio,

The eliminant of (1) and (3) is

$$\begin{vmatrix} a, & b, & c, & 0 \\ 0, & a, & b, & c \\ ay^2, & by, & e, & 0 \\ 0, & ay^2, & by, & e \end{vmatrix}=0$$

or for $a \neq 0$,

$$(4) \qquad ac^2y^4 - b^2cy^3 + y^2(b^2c - 2ace + b^2e) - b^2ey + ae^2 = 0.$$

The eliminant of (2) and (4) is

$$\begin{vmatrix} ac^2 & -b^2c & b^2c-2ace+b^2e & -b^2e & ae^2 & 0 \\ 0 & ac^2 & -b^2c & b^2c-2ace+b^2e & -b^2e & ae^2 \\ a & b & d & 0 & 0 & 0 \\ 0 & a & b & d & 0 & 0 \\ 0 & 0 & a & b & d & 0 \\ 0 & 0 & 0 & a & b & d \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} a(c^2d-b^2c+2ace-b^2e)-b^2c(b+c), & b(abc-bcd-c^2d), & -a^2e^2 \\ a & b & d \\ bc(ac+bd), & ac^2d-(b^2cd-2acde+b^2de-a^2e^2), & be(ae+bd) \end{vmatrix} = 0.$$

II. Solution by the PROPOSER.

To avoid the introduction of determinants of high order, we proceed thus: Multiply the third equation by a and replace $a^2x^2y^2$ by $ax^2.ay^2$ obtained from the first and second.

$\therefore b(ax+bx+c)y+bdx+ae+cd=0.$

Substituting in the second equation the value of y thus rationally determined, and dropping the factor a (the case $a=0$ being trivial), we obtain a second quadratic for x:

$$(a+b+d)b^2dx^2 + (2ade + 2cd^2 + bcd - abe - b^2e)bx + (ae+cd)^2 - b^2ce = 0.$$

The eliminant may now be determined in simple form.

186. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Eliminate x and y from the equations (1) $ax^3+bx^2+cx+d=0$, (2) $ay^3+by^2+cy+e=0$, (3) $ax^3y^3+bx^2y^2+cxy+f=0$, the eliminant to be rational in d, e, f.

Solved by H. F. MacNEISH, A. B., Instructor in University High School, Chicago, Ill., and G. B. M. ZERR A. M., Ph. D., Parsons, W. Va.

Using the same method as in No. 185.

GEOMETRY.

203. Additional solutions of problem 203 have been received from *G. W. GREENWOOD*, B.A. (Oxon) Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill., and *J. CHARLES RATHBUN*, A. B., Assistant in Physics, University of Washington.

205. Solutions of problem 205 have also been received from *G. B. M. ZERR*, A.M., Ph.D., Parsons, W. Va., and *G. W. GREENWOOD*, B. A. (Oxon) Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

206. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

$ABCD$ is circumscribed by a circle center O, and it circumscribes a circle radius r. The perpendiculars from C on the sides are x, y, z, u. Show that $\frac{1}{2}AC.BD=r\Sigma x$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The problem should read "the perpendiculars from O" instead of "the perpendiculars from C.

Let a, b, c, d denote the sides AB, BC, CD, DA, respectively; x the perpendicular on a; y, on b; z, on c; u, on d; R=circum-radius.

Then $x=\sqrt{(R^2-\frac{1}{4}a^2)}$. Now $R=AC/2\sin B$.

$AC^2=a^2+b^2-2ab\cos B=c^2+d^2+2cd\cos B$.

$$\therefore x=\frac{1}{2\sin B}\sqrt{(AC^2-a^2\sin^2 B)}=\frac{b-a\cos B}{2\sin B},\quad y=\frac{a-b\cos B}{2\sin B},$$

$$z=\frac{d+c\cos B}{2\sin B},\quad u=\frac{c+d\cos B}{2\sin B}.\quad \therefore r\Sigma x=\frac{r(a+b+c+d)-r(a+b-c-d)\cos B}{2\sin B},$$

$$r=\frac{2\sqrt{(abcd)}}{a+b+c+d},\quad a+c=b+d,\quad \cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

$$\therefore \cos B=\frac{(a-c)(a+c)+(b-d)(b-d)}{2(ab+cd)}=\frac{(a+c)(a+b-c-d)}{2(ab+cd)}.$$

$$\therefore r\Sigma x=\frac{\sqrt{(abcd)}}{\sin B}-\frac{\sqrt{(abcd)}.(a+b-c-d)^2}{4(ab+cd)\sin B},\quad \sin B=\frac{2\sqrt{(abcd)}}{ab+cd},$$

$4(ab+cd)\sin B=8\sqrt{(abcd)}$.

$$\therefore r\Sigma x=\frac{ab+cd}{2}-\frac{(a+b-c-d)^2}{8}.\text{ But } d=a+c-d.$$

$$\therefore r\Sigma x=\frac{ab+c(a+c-b)}{2}-\frac{(b-c)^2}{2}=\frac{ab+ac+bc-b^2}{2}$$

$$=\frac{ac+b(a+c)-b^2}{2}=\frac{ac+bd}{2}=\tfrac{1}{2}AC.BD.$$

207. Proposed by W. W. HART, University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into n equal parts by ruler and compass only, when n is a prime of the form 22^p+1.

The following construction gives good partial results for n equals *any* integer. If AB is the diameter of the circle, and C is the vertex of the equilateral triangle ABC, and if D is a point on AB at the distance $2AB/n$ from A, then draw the line CD cutting the circle at E and F; E being the more remote from

C. $AE=1/n$ circumference approximately. For low values of n this method is very practical; is it practical in general? How great is the error?

I. Solution by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Join OE. Let $\angle ACD=x$ and $\angle AOE=y$; then $DCB=60^\circ-x$; $ADC=120^\circ-x$; $DAE=90^\circ-\frac{1}{2}y$; $AED=30^\circ-x+\frac{1}{2}y$. $AD=2AB/n=4r/n$; $AB=AC=BC=2r$; $AO=OE=r$.

In $\triangle ADC$: $\dfrac{\sin x}{\sin(120-x)}=\dfrac{AD}{AC}=\dfrac{4r/n}{2r}=\dfrac{2}{n}$.

$\therefore \frac{1}{2}n\sin x=\sin(120-x)=\frac{1}{2}\sqrt{3}\cos x+\frac{1}{2}\sin x$.....(1). $(n-1)\sin x=\sqrt{3}\sqrt{(1-\sin^2 x)}$.

$\therefore \sin x=\sqrt{\dfrac{3}{n^2-2n+4}}$.....(2). $\therefore \sin(120-x)=\dfrac{n}{2}\sqrt{\dfrac{3}{n^2-2n+4}}$.....(3),

and $\cos(120-x)=\dfrac{n-4}{2\sqrt{(n^2-2n+4)}}$.....(4).

In $\triangle AOE$: $\dfrac{AE}{OE}=\dfrac{\sin y}{\sin(90-\frac{1}{2}y)}=\dfrac{\sin y}{\cos\frac{1}{2}y}$. $\therefore AE=\dfrac{r\sin y}{\cos\frac{1}{2}y}$.....(5).

In $\triangle AEC$: $\dfrac{AE}{AC}=\dfrac{\sin x}{\sin(30-x+\frac{1}{2}y)}$. $\therefore AE=\dfrac{2r\sin x}{\sin(30-x+\frac{1}{2}y)}$.....(6).

From (5) and (6), $\dfrac{\sin y}{\cos\frac{1}{2}y}=\dfrac{2\sin x}{\sin(30-x+\frac{1}{2}y)}$.

$\therefore \sin y[\sin(30-x)\cos\frac{1}{2}y+\sin\frac{1}{2}y\cos(30-x)]=2\sin x\cos\frac{1}{2}y$,

or $\sin y[-\cos(120-x)\cos\frac{1}{2}y+\sin\frac{1}{2}y\sin(120-x)]=2\sin x\cos\frac{1}{2}y$.

Hence from (1), $\sin y[-\cos(120-x)\cos\frac{1}{2}y+\frac{1}{2}n\sin\frac{1}{2}y\sin x]=2\sin x\cos\frac{1}{2}y$

or $\sin x[n\sin y\sin\frac{1}{2}y-4\cos\frac{1}{2}y]=2\sin y\cos\frac{1}{2}y\cos(120-x)$

or since $\sin y=2\sin\frac{1}{2}y\cos\frac{1}{2}y$, $\sin x[n\sin^2\frac{1}{2}y-2]=\sin y\cos(120-x)$.

Then from (2) and (4),

$$\frac{\sqrt{3}}{\sqrt{(n^2-2n+4)}}[n\sin^2\tfrac{1}{2}y-2]=\frac{n-4}{2\sqrt{(n^2-2n+4)}}\sin y.$$

$\therefore 2\sqrt{3}\left[\dfrac{n(1-\cos y)}{2}-2\right]=(n-4)\sin y$. $\therefore 3(n-n\cos y-4)^2=(1-\cos^2 y)(n-4)^2$.

$$\therefore \cos y=\frac{(n-4)[3n\pm\sqrt{(n^2+16n-32)}]}{4(n^2-2n+4)}.$$

Then for the positive value of the radical we obtain the following values of y for $n=3, 4, 5, \ldots$

n	$\cos y$	$\log \cos y$	y	$2\pi/n$	Error	Rate of error
3	$-\frac{1}{2}$	——	120°	120°	0	.0000
4	0	——	90°	90°	0	.0000
5	——	9.49107	$71^\circ\ 57'\ 12''$	72°	$2'\ 48''$	.0007
6	$\frac{1}{2}$	——	60°	60°	0	.0000
7	——	9.79393	$51^\circ\ 31'\ 23''$	$51^\circ\ 25'\ 43''$	$5'\ 40''$	.0018
8	——	9.84806	$45^\circ\ 11'\ 14''$	45°	$11'\ 14''$	.0042
9	——	9.88248	$40^\circ\ 16'\ 38''$	40°	$16'\ 38''$	.0069
10	——	9.90599	$36^\circ\ 21'\ 18''$	36°	$21'\ 18''$	.0099
11	——	9.92286	$33^\circ\ 8'\ 53''$	$32^\circ\ 43'\ 38''$	$25'\ 15''$	.0129
12	——	9.93545	$30^\circ\ 28'\ 15''$	30°	$28'\ 15''$	.0157
24	——	9.98363	$15^\circ\ 38'$	15°	$38'$	.0422
48	——	9.99516	$8^\circ\ 32'\ 30''$	$7^\circ\ 30'$	$1^\circ\ 2'\ 20''$	.1389
90	——	9.99818	$5^\circ\ 14'\ 30''$	4°	$1^\circ\ 14'\ 30''$	.3104
180	——	9.99967	$2^\circ\ 15'$	2°	$15'$	.1250
360	——	9.99995	$49'\ 40''$ to $54'\ 40''$	1°	$7'\ 30''$	.1250

The construction therefore has an error of over $1\frac{1}{2}\%$ for values of $n>12$, and for large values of n the error is very great.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let O be the center of the circle, $AO=R=EO$. Then $CO=R\sqrt{3}$, $DO=R-4R/n=(R/n)(n-4)$. $\therefore$ tangent $DCO=(n-4)/n\sqrt{3}$.

$$\sin DCO=\frac{n-4}{\sqrt{(4n^2-8n+16)}},\quad \cos DCO=\frac{n\sqrt{3}}{\sqrt{(4n^2-8n+16)}},$$

$$\sin CED=\frac{CO\sin DCO}{EO}=\frac{(n-4)\sqrt{3}}{\sqrt{(4n^2-84+16)}}.$$

$$\sin(90^\circ+DOE)=\sin(180^\circ-DEO-DCO)=\sin(DEO+DCO).$$

$$\therefore \cos DOE=\sin(DEO+DCO)=\frac{(n-4)[3n+\sqrt{(n^2+16n-32)}]}{4n^2-8n+16}.$$

For $n=3$, 4 and 6 the error is nothing.
For $n=5$ the side and angle are a trifle small.
For $n>6$ the side and angle are too large but the error varies.
For $n=8$, $\cos DOE=.70479$, $DOE=45^\circ\ 11'\ 14.5''$, an error of $11'\ 14.5''$.
For $n=12$, $\cos DOE=.86186$, $DOE=30^\circ\ 28'\ 25.7''$, an error of $28'\ 25.7''$.*
For $n=20$, $\cos DOE=.95091$, $DOE=18^\circ\ 2'\ 40''$, an error of only $2'\ 40''$.

*In solution I, Mr. MacNeish finds the error for $n=12$ to be $28'15''$, otherwise the two solutions agree. Ed.

For $n=72$, $\cos DOE=.99559$, $DOE=5°\ 29'$, an error of $29'$.

For large values of n the error is much too great for any purpose.

Also solved by *J. E. SANDERS*, Hackney, Ohio.

CALCULUS.

165. Proposed by CAPT. T. C. DICKSON, Ordnance Department, United States Army, Washington, D. C.

Solve by integration the differential equation

$$\frac{d^2\xi}{dt^2}+\frac{A}{B}\left(\frac{d\xi}{dt}\right)^2-\frac{C}{B}=0,$$

in which A, B, C are given functions of ξ, but independent of t.

Solution by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

In view of the nature of the coefficients, we regard ξ as the independent variable and t the dependent, the formulae of transformation being

$$\frac{d\xi}{dt}=1\div\frac{dt}{d\xi},\quad \frac{d^2\xi}{dt^2}=-\frac{d^2t}{d\xi^2}\div\left(\frac{dt}{d\xi}\right)^3.$$

The given equation thus becomes

$$\frac{d^2t}{d\xi^2}-\frac{A}{B}\frac{dt}{d\xi}+\frac{C}{B}\left(\frac{dt}{d\xi}\right)^3=0.$$

Set $dt/d\xi=y$, whence $t=\int yd\xi$. Then $\dfrac{dy}{d\xi}-\dfrac{A}{B}y+\dfrac{C}{B}y^3=0.$

Divide by y^3 and set $z=y^{-2}$. The resulting differential equation

$$\frac{dz}{d\xi}+\frac{2A}{B}z-\frac{2C}{B}=0$$

is linear. By the usual method, we get

$$z=2e^{-\lambda}\left(\int\frac{C}{B}e^{\lambda}\,d\xi+k\right),\quad \lambda\equiv2\int\frac{A}{B}d\xi.\quad \therefore\ t=\int z^{-\frac{1}{2}}d\xi.$$

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the value of y from the Eulerian equation

$$y=\int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}.$$

Solution by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

Let $x=\dfrac{\sqrt{3}(z+1)}{z-1}$, $z^3-1=u^3$.

$$\therefore \int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}=-\int\frac{dz}{z\sqrt[3]{[4(z^3-1)}}=-\int\frac{udu}{\sqrt[3]{4(u^3+1)}}.$$

$$\therefore\ y=\frac{1}{3\sqrt[3]{4}}\int\frac{du}{1+u}-\frac{1}{6\sqrt[3]{4}}\int\frac{(2u-1)du}{1-u+u^2}-\frac{1}{2\sqrt[3]{4}}\int\frac{du}{1-u+u^2}.$$

$$\therefore\ y=\frac{1}{6\sqrt[3]{4}}\log\left(\frac{(1+u)^2}{1-u+u^2}\right)-\frac{1}{\sqrt{3}\sqrt[3]{4}}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right),$$

where $u=\dfrac{\sqrt[3]{[6\sqrt{3}(x^2+1)]}}{x-\sqrt{3}}$.

170. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find the center-locus of conics having 4-point contact with a given conic at a given point. Show that the conic of minimum eccentricity is given by $e^4\tan^2\varphi+4e^2-4=0$, where e is its eccentricity, and φ is the angle which the linear center-locus above makes with the normal to the curve at the point.

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the State University, Athens, Ohio.

The coördinates of the center of any conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ \text{.....}(1),\ \text{are}$$

$$x_1=\frac{hf-bg}{ab-h^2}\text{.....}(2),\quad y_1=\frac{gh-af}{ab-h^2}\text{.....}(3),$$

and the eccentricity is given by

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0\text{.....}(4).$$

If the tangent and normal to the given curve be the axes of abscissas and ordinates, the equation of the conic having 4-point contact with the given conic is of the form

$$ax^2+2hxy+by^2+2gx-\lambda x^2=0\text{.....}(5),\ \text{or},\ (a-\lambda)x^2+2hxy+by^2+2gx=0\text{.....}(6).$$

Comparing (1) and (6), $a=a-\lambda$, $f=0$, $c=0$.....(7), and (2) and (3) become

$$x_1=-\frac{bg}{(a-\lambda)b-h^2}\text{.....}(8),\quad y_1=\frac{gh}{(a-\lambda)b-h^2}\text{.....}(9).$$

(9)÷(8) gives $y_1-(h/b)x_1$.....(10), the locus-center. Thus the "slope" is

$-(h/b)$, and the tangent of the angle (10) makes with the y axis is $\tan_\theta=-(b/h)$.....(11). Substituting $a=a-\lambda$ in (4), we find

$$u=\frac{1}{e^2}=\frac{a-\lambda+b}{\sqrt{[(a-\lambda-b)^2+4h^2}}+\tfrac{1}{2}.....(12).$$

Equating $du/d\lambda$ to zero, we find $(a-\lambda)b-b^2-2h^2=0$, or,

$$a-\lambda=\frac{b^2+2h^2}{b}.....(13).\quad \text{Then } a-\lambda-b=\frac{2h^2}{b}.....(14).$$

Substituting (7), (13) and (14) in (4) and reducing

$$e^4+\frac{4h^2}{b^2}(e^2-1)=0, \text{ or } \frac{b^2}{h^2}e^4+4e^2-4=0.....(15).$$

This becomes by (11), $e^4\tan^2\varphi+4e^2-4=0$.....(16).

Also solved by *G. W. GREENWOOD*, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., and by *G. B. M. ZERR*, A. M., Ph. D., Parsons, W. Va.

171. Proposed by J. E. SANDERS, Hackney, Ohio.

A thread passes spirally around a *rough* cylinder 10 feet high and 6 inches in diameter. How far will a pigeon fly in unwinding the thread if the distance between the coils is 4 inches, and the thread *unwound* is at all times *horizontal*?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let r=radius of cylinder=3 inches=$\frac{1}{4}$ feet; d=distance between coils =$\frac{1}{3}$ feet; m=number of coils=30. Then from Vol. I, No. 6, pages 318-320 of this Journal* we have for the required distance

$S=\pi m^2\sqrt{(4\pi^2r^2+d^2)}$ feet$=900\pi\sqrt{(\frac{1}{4}\pi^2+\frac{1}{9})}$ feet$=4540.246$ feet, nearly.

II. Solution by H. B. LEONARD, B. S.

Circumference$=6\pi$. Length of thread on one turn$=\sqrt{[(6\pi^2)+4^2]}$. Number of turns$=(12\times10)\div4=30$. Total rotation$=60\pi$. Angle of elevation of thread on cylinder$=a=\sin^{-1}\dfrac{4}{\sqrt{[(6\pi^2)+4^2]}}$. During the bird's flight, an unwinding of $d\theta$ produces an increase of $dz=3\tan a.d\theta$ in altitude, of $dr=3\sec a.d\theta$ in direction of flight, of $dc=3\sqrt{(\theta^2\sec^2a+1)}d\theta$ normal to direction of flight.

$$(ds)^2=(dr)^2+(dz)^2+(dc)^2=(9\sec^2a+9\tan^2a+9\theta^2\sec^2a+9)(d\theta)^2$$
$$=(18\sec^2a+9\theta^2\sec^2a)(d\theta)^2.\quad ds=3\sec a\sqrt{(2+\theta^2)}d\theta.$$

$$S=\int_0^{60\pi}3\sec a\sqrt{(2+\theta^2)}d\theta=3\sec a\int_0^{60\pi}\sqrt{(2+\theta^2)}d\theta=3\sec a\Big[\tfrac{1}{2}\{\theta\sqrt{(2+\theta^2)}$$

* See also Vol. 1, pp. 88-89. ED.

$$+2\log[\theta+\sqrt{(2+\theta^2)}]\}\Big]_{\theta=0}^{\theta=60}=3\sec\alpha\{\tfrac{1}{2}[60\pi\sqrt{(2+3600\pi^2)}$$
$$+2\log[60\pi+\sqrt{(2+3600\pi^2)}]-\tfrac{1}{2}[0\sqrt{2}+2\log(0+\sqrt{2})]\}$$
$$=3\sec\alpha\{30\pi\sqrt{(2+3600\pi^2)}+\log[60\pi+\sqrt{(2+3600\pi^2)}]-\log\sqrt{2}\}=4540 \text{ feet.}$$

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ... denote all its distinct divisors, then $2^n>2[n+1][n'+1][n''+1]$.....

Solution by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

There is a single exception $n=3$, for which $2^3=2[3+1]$. The corrected theorem may be proved by induction, using the following lemma:

If p is an odd prime number and d and π positive integers,

$$[1+d]^{p^\pi}>[1+d][1+pd][1+p^2d].....[1+p^\pi d],$$

except for $d=1$, $\pi=1$, $p=3$, the equality sign then holding.

For proof we apply $p^\pi-1=p-1+p[p-1]+p^2[p-1]+.....+p^{\pi-1}[p-1]$.

$$\therefore [1+d]^{p^\pi}=[1+d][1+d]^{p-1}[1+d]^{p(p-1)}.....[1+d]^{p^{\pi-1}(p-1)}.$$

But $[1+d]^{t(p-1)}\overline{\gtrless}[1+td]^{p-1}\overline{\gtrless}1+[p-1]td+[td]^{p-1}\overline{\gtrless}1+[p-1]td+td$, if $p>2$. The equality signs hold simultaneously only when $t=d=1$, $p=3$. Hence, for $p>2$, $[1+d]^{t(p-1)}>1+ptd$ unless $t=d=1$, $p=3$, so that the lemma follows.

To prove the theorem by induction, we note that it is true for $n=p^\pi>3$, in view of the lemma for $d=1$. Assume that it has been verified for $n=p_1^{\pi_1}p_2^{\pi_2}$..... We proceed to prove it true for $N=np^\pi$, p being prime to n. We have

$$2^N=[2^n]^{p^\pi}\overline{\gtrless}[(1+1)(1+n)(1+n').....]^{p^\pi}$$

$$\overline{\gtrless}[(1+1)(1+p).....(1+p^\pi)][(1+n)(1+pn)(1+p^2n).....][(1+n')(1+pn').....],$$

in view of the lemma. But the distinct divisors of N are

$$1,\ n,\ n',\ .....,\ p,\ pn,\ pn',\ .....,\ p^2,\ p^2n,\ .....,\ p^\pi,\ p^\pi n,\ .....$$

The theorem is therefore true for N.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least integral numbers such that their sum shall be a square and the sum of their squares a biquadrate.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x and y be the numbers, then for $x+y=1$, $x^2+y^2=13^4$,
$x=120$, $y=-119$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

189. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

$$\text{Solve } 3x-y=xy-x^2,\ \sqrt[3]{y}=\sqrt{\frac{2y-x}{2}}.$$

GEOMETRY.

212. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Given two triangles ABC and $A'B'C'$ lying in the same plane. The side $B'C'$ cuts the sides AC, BC, and AB in the points I, H, and G, respectively; the side $A'B'$ cuts the same sides, AC, BC, and AB in D, F, and E, respectively; and $A'C'$ cuts AC, BC, and AB in M, L, and K, respectively. Prove that

$$(DA'.EA'.A'F)\ (GB'.HB'.B'I)\ (MC'.LC'.C'K)$$
$$=-(KA'.A'L.A'M)(FB'.B'E.B'D)\ (IC'.C'H.C'G).$$

213. Proposed by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Construct an equilateral triangle which shall have its vertices in three given parallel lines.

214. Proposed by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Inscribe in a given circle a triangle whose sides shall pass through three given points.

CALCULUS.

173. Proposed by J. E. SANDERS, Hackney, O.

Find the area of the greatest ellipse that can be inscribed in the quadrant of a given circle.

MECHANICS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

P balances W on a system of n movable pulleys of equal weight, each hanging by a separate string. If P is moved find the maximum acceleration of W.

AVERAGE AND PROBABILITY.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the aver-area of all the segments possible?

151. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If A and B play tennis together, find the probability that A will win, given that a is the probability that A will win a given point and b the the probability that B will win the point.

NOTES.

The index for Vol. X will be mailed with the opening number for 1904.

Mr. C. Gilman has been appointed Instructor in Mathematics and Surveying at Harvard University.

Mr. Walter J. Risley has been appointed Instructor in Mathematics at the Armour Institute, Chicago.

Mr. C. R. Burger has been elected Assistant Professor of Mathematics at the Colorado School of Mines.

Dr. H. E. Hawkes has been promoted to an Assistant Professorship of Mathematics at Yale University.

Professor R. J. Aley, of the University of Indiana, has been elected to the editorship of the *Educator-Journal.*

Professor F. Anderegg, of Oberlin College, is away on leave of absence. He is spending the year in study and sightseeing in his native country, Switzerland.

Dr. Charles H. Ashton, previously Instructor in Mathematics at Harvard University, has been appointed Assistant Professor of Mathematics at the University of Kansas.

The Chicago Section of the American Mathematical Society will meet at St. Louis, Mo., during the Christmas recess.

The San Francisco Section of the American Mathematical Society met at the University of California on December 19, 1903.

The Central Association of Science and Mathematics Teachers held its annual meeting in Chicago on November 27 and 28, 1903.

An organization meeting of Teachers of Mathematics in ths Middle States and Maryland was held in affiliation with the Association of Colleges and Preparatory Schools of the Middle States and Maryland at Columbia University, on Saturday, November 28th, 1903, in Milbank Memorial Chapel, Teachers College.

Any of our subscribers wishing to correspond with Mr. Finkel on matters of business pertaining to the MONTHLY should send such correspondence directly to him at 204 St. Marks Square, Philadelphia, Pa. All subscriptions and requests for sample copies, etc., should be sent to W. C. Calland, Treasurer, Drury College, Springfield, Mo.